Illustrating the two-phase method

Example 1 *We use the two-phase method to solve the linear programming problem*

$$\begin{array}{ll} \textit{maximise} & z = 3x_1 + x_2 \\ \textit{subject to} & x_1 + x_2 + s_1 = 6 \\ & 4x_1 - x_2 - s_2 = 8 \\ & 2x_1 + x_2 = 8 \\ & x_1, x_2, s_1, s_2 \geq 0. \end{array} \tag{1}$$

In ***Phase I*** *we solve the auxiliary linear programming problem*

$$\begin{array}{lrcl} \textit{maximise} & z' = -u_1 - u_2 & & \\ \textit{subject to} & x_1 + x_2 + s_1 & = & 6 \\ & 4x_1 - x_2 - s_2 + u_1 & = & 8 \\ & 2x_1 + x_2 + u_2 & = & 8 \\ & x_1, x_2, s_1, s_2, u_1, u_2 \geq 0. & & \end{array}$$

to obtain a basic feasible solution to the original problem, (1). In ***Phase II*** *we solve the original problem.*

We use $\{s_1, u_1, u_2\}$ *as our initial basis with basic feasible solution* $s_1 = 6$, $u_1 = 8$ *and* $u_2 = 8$ *with the remaining variables set to zero. To proceed, we want to write our objective function* z' *as a function of the non-basic variables. To do this note that we have*

$$4x_1 - x_2 - s_2 + u_1 = 8 \tag{2}$$

$$2x_1 + x_2 + u_2 = 8 \tag{3}$$

$$z' + u_1 + u_2 = 0 \tag{4}$$

so that subtraction of each row containing an artificial variable, from the z' *row will eliminate the artificial variables from this equation, that is we perform (4) - (3) - (2) to obtain*

$$z' - 6x_1 + s_2 = -16.$$

Note that our original objective function $z = 3x_1 + x_2$ *does not contain any of the basis variables,* $\{s_1, u_1, u_2\}$, *so we have* $z - 3x_1 - x_2 = 0$. *We are ready to commence Phase I.*

Phase I

We create the initial simplex tableau.

T_1	z'	z	x_1	x_2	s_1	s_2	u_1	u_2	
s_1	0	0	1	1	1	0	0	0	6
u_1	0	0	4	-1	0	-1	1	0	8
u_2	0	0	2	1	0	0	0	1	8
II	0	1	-3	-1	0	0	0	0	0
I	1	0	-6	0	0	1	0	0	-16

We proceed as usual for the simplex method using z' *as the objective but performing row operations on the row labelled* II *which corresponds to the objective function* z *to ensure that it is always expressed as a function of non-basic variables only. From the bottom row, we see that introducing* x_1 *into the basis will increase* z' *and* $\theta_1 = \min\{6, 8/4, 8/2\} = 2$ *so that we push out* u_1. *We perform row operations on* T_1 *to introduce* x_1 *into the basis, obtaining the tableau* T_2.

T_2	z'	z	x_1	x_2	s_1	s_2	u_1	u_2	
s_1	0	0	0	$5/4$	1	$1/4$	$-$	0	4
x_1	0	0	1	$-1/4$	0	$-1/4$	$-$	0	2
u_2	0	0	0	$3/2$	0	$1/2$	$-$	1	4
II	0	1	0	$-7/4$	0	$-3/4$	$-$	0	6
I	1	0	0	$-3/2$	0	$-1/2$	$-$	0	-4

Note that we do not calculate the values corresponding to u_1 as we do not intend to reintroduce this variable. We now remove u_2 and can introduce either x_2 or s_2. We introduce x_2 and, performing row operations on T_2, obtain the tableau T_3.

T_3	z'	z	x_1	x_2	s_1	s_2	u_1	u_2	
s_1	0	0	0	0	1	$-1/6$	$-$	$-$	$2/3$
x_1	0	0	1	0	0	$-1/6$	$-$	$-$	$8/3$
x_2	0	0	0	1	0	$1/3$	$-$	$-$	$8/3$
II	0	1	0	0	0	$-1/6$	$-$	$-$	$32/3$
I	1	0	0	0	0	0	$-$	$-$	0

We have now moved u_1 and u_2 out of the basis. We have $z' = 0$ and have a basic feasible solution $(x_1, x_2, s_1, s_2)^T = (8/3, 8/3, 2/3, 0)^T$.[1]

Phase II

We can reduce T_3 to remove reference to the Phase I variables.

T_3	z	x_1	x_2	s_1	s_2	
s_1	0	0	0	1	$-1/6$	$2/3$
x_1	0	1	0	0	$-1/6$	$8/3$
x_2	0	0	1	0	$1/3$[8]	$8/3$
	1	0	0	0	$-1/6$	$32/3$

Introducing s_2 into the basis will improve the objective. We remove x_2 from the basis to obtain the tableau T_4.

T_4	z	x_1	x_2	s_1	s_2	
s_1	0	0	$1/2$	1	0	2
x_1	0	1	$1/2$	0	0	4
s_2	0	0	3	0	1	8
	1	0	$1/2$	0	0	12

Reading the bottom row we have $z = 12 - \frac{1}{2}x_2$. The basic feasible solution $(x_1, x_2, s_1, s_2)^T = (4, 0, 2, 8)^T$ is optimal with $z = 12$.

[1] It can be readily checked that we have not made a mistake in our algebra by checking that (1) is satisfied at this solution: $\frac{8}{3} + \frac{8}{3} + \frac{2}{3} = 6$, $4\frac{8}{3} - \frac{8}{3} = 8$ and $2\frac{8}{3} + \frac{8}{3} = 8$. Also $z = 3\frac{8}{3} + \frac{8}{3} = \frac{32}{3}$ which verifies our final equation.