Example of the Simplex Algorithm

Consider our original motivating example concerning a furniture company producing oak chairs and oak tables from its available resources.

$$(\mathrm{S})\qquad \begin{array}{lrcl} \text{maximise} & z & = & x_1 + 2x_2 \\ \text{subject to} & 5x_1 + 20x_2 & \leq & 400 \\ & 10x_1 + 15x_2 & \leq & 450 \\ & x_1, x_2 & \geq & 0. \end{array}$$

There are four extreme points $(0,0)$, $(0,20)$, $(24,14)$ and $(45,0)$ and the optimal $z = 52$ occurs at the extreme point $(24,14)$. In canonical form the problem is

$$(\mathrm{C})\qquad \begin{array}{lrcl} \text{maximise} & z & = & x_1 + 2x_2 \\ \text{subject to} & 5x_1 + 20x_2 + s_1 & = & 400 \\ & 10x_1 + 15x_2 + s_2 & = & 450 \\ & x_1, x_2, s_1, s_2 & \geq & 0. \end{array}$$

On Question 2 of Question Sheet Two you showed that the canonical problem has four basic feasible solutions $(0,0,400,450)$, $(0,20,0,150)$, $(24,14,0,0)$ and $(45,0,175,0)$.

- Note the one-to-one mapping of the extreme points of the canonical problem to the extreme points of the corresponding problem in standard form.
- If you want to prove this see Question 5 of Question Sheet Three.
- If we have a problem in standard form (S) and solve the associated problem in canonical form (C) then the restriction of the optimal solution/extreme points of (C) to the variables in (S) corresponds to the optimal solution/extreme points of (S). (See also Question 5 of Question Sheet One.)

Let's consider a simplex algorithm approach to this problem. We will make use of the two fundamental equations

$$\begin{aligned} \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N &= \mathbf{B}^{-1}\mathbf{b}, \\ z &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \left(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x}_N. \end{aligned}$$

Step One: Initial BFS, basis $\{s_1, s_2\}$.

T_1	z	x_1	x_2	s_1	s_2	
s_1	0	5	20	1	0	400
s_2	0	10	15	0	1	450
	1	-1	-2	0	0	0

- Basic feasible solution is $(0,0,400,450)$ with $z = 0 + x_1 + 2x_2$.
- All of the reduced costs are positive: introducing either x_1 or x_2 into the basis will increase z.
- Let's choose to introduce x_2 into the basis.

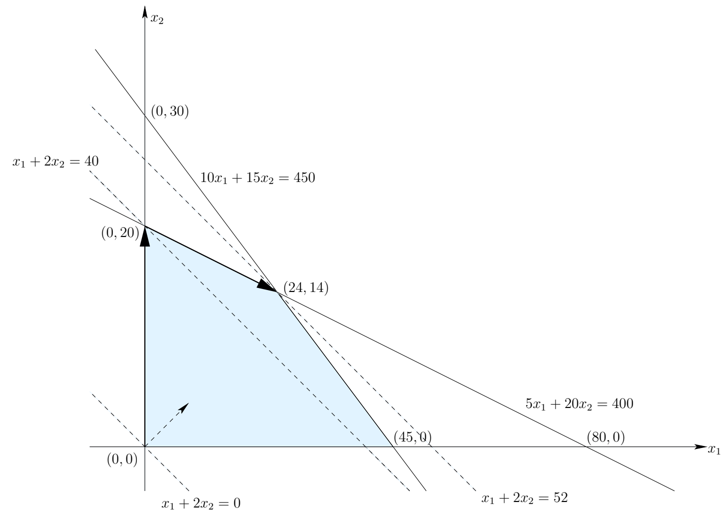

Figure 1: Representation of the simplex algorithm for the original standard problem. Step One corresponds to the extreme point $(0,0)$. In Step Two we introduce x_2 into the basis which increases the objective function. Pushing out s_1 corresponds to moving to $(0,20)$ which is feasible and the move we make; pushing out s_2 corresponds to moving to $(0,30)$ which is not feasible. In Step Three we introduce x_1 into the basis. Pushing out s_2 corresponds to moving to $(24,14)$ which is feasible and optimal and the move we make; pushing out x_2 corresponds to moving to $(80,0)$ which is not feasible.

Step Two: Changing the basis, introducing x_2.

T_1	z	x_1	x_2	s_1	s_2	
s_1	0	5	20	1	0	400
s_2	0	10	15	0	1	450
	1	-1	-2	0	0	0

$\rightsquigarrow$

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	$1/4$	1	$1/20$	0	20
s_2	0	$25/4$	0	$-3/4$	1	150
	1	$-1/2$	0	$1/10$	0	40

- To maintain feasibility, replace s_1 by x_2 in the basis
- Basic feasible solution is $(0, 20, 0, 150)$ with $z = 40 + \frac{1}{2}x_1 - \frac{1}{10}s_1$.
- There is a positive reduced cost: introducing x_1 into the basis will increase z.

Step Three: Changing the basis, introducing x_1.

T_2	z	x_1	x_2	s_1	s_2	
x_2	0	$1/4$	1	$1/20$	0	20
s_2	0	$25/4$	0	$-3/4$	1	150
	1	$-1/2$	0	$1/10$	0	40

$\rightsquigarrow$

T_3	z	x_1	x_2	s_1	s_2	
x_2	0	0	1	$2/25$	$-1/25$	14
x_1	0	1	0	$-3/25$	$4/25$	24
	1	0	0	$1/25$	$2/25$	52

- To maintain feasibility, replace s_2 by x_1 in the basis
- Basic feasible solution is $(24, 14, 0, 0)$ with $z = 52 - \frac{1}{25}s_1 - \frac{2}{25}s_2$.
- All the reduced costs are negative: this is the optimal solution.