# MA20226 - Question Sheet Five

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**Set:** Tutorials of Week 8, commencing Monday 21st November 2011.
**Due in:** Practicals of Week 9, commencing Monday 28th November 2011. You should write both your name and tutorial group on your work. If you are unable to make your practical then you should ensure that you hand the work to me personally before the practical is held; my office is 4W4.10.
**Task:** Attempt questions 1-3; questions 4-7 are additional questions and may be used for tutorial discussion.

1. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu = 0, \sigma^2)$ random quantities and we wish to test the hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{versus} \quad H_1 : \sigma^2 = \sigma_1^2$$

where $\sigma_1^2 < \sigma_0^2$.

   (a) Use the Neyman-Pearson lemma to find the test of significance $\alpha$ which has the largest power. Try to ensure that your final test statistic is as simple as possible.

   (b) What is the sampling distribution of your test statistic? Hence find the critical value for a test of significance $\alpha = 0.05$ (you do not need to evaluate this critical value, but express it in terms of a quantile of the distribution of your test statistic).

2. Suppose $X_1, \ldots, X_{10}$ are independent and identically distributed $N(\mu, \sigma^2 = 25)$ random quantities.

   (a) Carry out the following one-sided test by stating a test statistic and finding the critical value of the test:

$$H_0 : \mu = 105 \quad \text{versus} \quad H_1 : \mu > 105$$

   (b) Carry out this test at the 5% significance level when a sample mean $\overline{x} = 108$ is obtained.

   (c) Evaluate the power of this test at the points $\mu = 106$ and $\mu = 110$.

3. Again suppose $X_1, \ldots, X_{10}$ are independent and identically distributed $N(\mu, \sigma^2 = 25)$ random quantities.

   (a) Carry out the following two-sided test by stating a test statistic and finding the critical value of the test:

$$H_0 : \mu = 105 \quad \text{versus} \quad H_1 : \mu \neq 105$$

(b) Carry out this test at the 5% significance level when a sample mean $\overline{x} = 108$ is obtained.

(c) Evaluate the power of this test at the points $\mu = 106$ and $\mu = 110$.

(d) Would you have expected the power to be greater or smaller than that of the one-sided test?

4. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $Exp(\lambda)$ random quantities, so $f(x|\lambda) = \lambda \exp(-\lambda x)$. We wish to test the hypotheses

$$H_0 : \lambda = \lambda_0 \quad \text{versus} \quad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$.

(a) Use the Neyman-Pearson lemma to find the test of significance $\alpha$ with the largest power. Try to ensure that your final test statistic is as simple as possible.

(b) If you knew the sampling distribution of your test statistic, how would you find the critical value for a test of significance $\alpha = 0.05$? You do not need to evaluate this critical value, but express it in terms of a quantile of the distribution of your test statistic.

5. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, 1)$ random quantities, and you want a test with significance level 5% of the one-sided hypotheses

$$H_0 : \mu = 0 \quad \text{versus} \quad H_1 : \mu < 0$$

(a) Find the critical value $k$ such that the null hypothesis will be rejected if $\overline{x} \leq k$ (this critical value will depend on the sample size through the variance of $\overline{X}$).

(b) Suppose that you require your test to have power of at least 0.95 when $\mu$ is actually equal to -0.25. What is the probability that $H_0$ is not rejected when $\mu = -0.25$ (as a function of $n$)? So how big must the sample size $n$ be to meet this power requirement?

6. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, \sigma^2)$ random quantities.

(a) What is the sampling distribution of $S^2$?

(b) Set up a hypothesis test to determine whether the variability of some process has changed: Test the hypothesis $H_0 : \sigma^2 = 10$ against $H_1 : \sigma^2 \neq 10$, when a sample of size 20 yields an observed value of $s^2 = 13.8$.

7. The Neyman-Pearson lemma states that of all tests with significance $\alpha$ of the simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

the test which uses the critical region

$$C^* \quad = \quad \{(x_1, \ldots, x_n) : \lambda(x_1, \ldots, x_n; \theta_0, \theta_1) \leq k\},$$

where $\lambda(x_1, \ldots, x_n; \theta_0, \theta_1) = f(x_1, \ldots, x_n \,|\, \theta = \theta_0)/f(x_1, \ldots, x_n \,|\, \theta = \theta_1)$ and $k$ is a constant chosen to ensure the significance is $\alpha$, is the one with the largest power. Denote $(X_1, \ldots, X_n)$ by $\underline{X}$ and $(x_1, \ldots, x_n)$ by $\underline{x}$.

(a) Show that for any set $A \subset C^*$,

$$P(\underline{X} \in A \,|\, \theta = \theta_0) \;\leq\; kP(\underline{X} \in A \,|\, \theta = \theta_1).$$

(b) Show that for any set $A \subset \overline{C^*}$ (the compliment of $C^*$)

$$P(\underline{X} \in A \,|\, \theta = \theta_0) \;>\; kP(\underline{X} \in A \,|\, \theta = \theta_1).$$

(c) Let $C$ be some other critical region with significance level $\alpha$. Show that

$$P(\underline{X} \in C^* \,|\, \theta) - P(\underline{X} \in C \,|\, \theta) \;=\; P(\underline{X} \in (C^* \cap \overline{C}) \,|\, \theta) - P(\underline{X} \in (C \cap \overline{C^*}) \,|\, \theta),$$

where $\overline{C}$ denotes the compliment of $C$.

(d) Hence show that

$$P(\underline{X} \in C^* \,|\, \theta = \theta_1) - P(\underline{X} \in C \,|\, \theta = \theta_1) \;\geq\; 0.$$

You have now proved the Neyman-Pearson lemma.