MA20226 - Question Sheet Four

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2011/12 Semester I

Set: Tutorials of Week 7, commencing Monday 14th November 2011.
Due in: Tutorials of Week 8, commencing Monday 21st November 2011. You should write both your name and tutorial group on your work. If you are unable to make your practical then you should ensure that you hand the work to me personally *before* the practical is held; my office is 4W4.10.
Task: Attempt questions 1-3; questions 4-5 are additional questions and may be used for tutorial discussion.

1. To estimate the mean gestation period of domestic dogs, 15 randomly selected dogs are observed during pregnancy. Their gestation periods, in days, are:

   62.0 61.4 59.8 62.2 60.3 60.4 59.4 60.2 60.4 60.8 61.8 59.2 61.1 60.4 60.9

   Letting X_i denote the observed gestation period of the ith dog, we assume that the X_is are iid $N(\mu, \sigma^2)$. We find the following data summaries: $\sum_{i=1}^{15} x_i = 910.3$ and $\sum_{i=1}^{15} x_i^2 = 55254.35$.

   (a) Evaluate a 99% confidence interval for μ when σ^2 is known to be 1.
   (b) Evaluate a 95% confidence interval for σ^2 (assumed unknown).
   (c) Evaluate a 99% confidence interval for μ when σ^2 is assumed unknown.
   (d) Comment on the differences or similarities of the two intervals for μ.

2. Low-birthweight (LBW) babies present special problems for their parents. Firstly, they are at risk of physical and development problems. Secondly, they can appear to be unresponsive and unpredictable. A sample of 56 LBW infants were selected and at six months old, their mental activity was measured using the Psychomotor Development Index (PDI) of the Bayley Scales of Infant Development. The data for the LBW infants on the PDI is given below.

96	125	89	127	102	112	120	108	92	120	104	89	92	89
120	96	104	89	104	92	124	96	108	86	100	92	98	117
112	86	116	89	120	92	83	108	108	92	120	102	100	112
100	124	89	124	102	102	116	96	95	100	120	98	108	126

   Letting X_i denote the PDI score of the ith infant, we assume that the X_is are iid $N(\mu, \sigma^2)$. The data may be summarised by $\sum_{i=1}^{56} x_i = 5831$ and $\sum_{i=1}^{56} x_i^2 = 615863$.

   (a) Evaluate a 90% confidence for σ^2. You may find the following table useful in which the value given is $\chi^2_{\nu,\alpha}$ where $P(\chi^2_\alpha > \chi^2_{\nu,\alpha}) = \alpha$ for the χ^2 distribution on ν degrees of freedom.

α / ν	0.99	0.975	0.95	0.5	0.1	0.05	0.025	0.01
55	33.570	36.398	38.958	54.335	68.796	73.311	77.380	82.292
56	34.350	37.212	39.801	55.335	69.919	74.468	78.567	83.513
57	35.131	38.027	40.646	56.335	71.040	75.624	79.752	84.733

(b) Evaluate a 95% confidence interval for μ. You may find the following table useful in which the value given is $t_{\nu,\alpha}$ where $P(t_\alpha > t_{\nu,\alpha}) = \alpha$ for Student's t-distribution on ν degrees of freedom.

	One-tailed Probabilities				
α / ν	0.25	0.05	0.025	0.01	0.005
55	0.679	1.673	2.004	2.396	2.668
56	0.679	1.673	2.003	2.395	2.667
57	0.679	1.672	2.002	2.394	2.665
∞	0.674	1.645	1.960	2.326	2.576

(c) A normative population mean of 100 is usually found with the PDI. Does the confidence interval you calculated in part 2(b) support this value?

3. Let $X_1, \ldots, X_n$ be independent and identically distributed exponential random quantities with expectation θ and variance θ^2. Hence, for each $i = 1, \ldots, n$, the corresponding probability density function is

$$f(x \mid \theta) \quad = \begin{cases} \frac{1}{\theta}\exp(-x/\theta) & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the maximum likelihood estimator of θ is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(b) Given that $Y = \frac{2n}{\theta}\overline{X}$ is a χ^2-distribution with $2n$ degrees of freedom, construct a $100(1-\alpha)\%$ confidence interval for θ.

4. Sometimes it is the case that a one-sided rather than a two-sided confidence interval is required, which means that we want the realisation of a random (half) interval of the form either

$$P\{\theta > g_1(X_1, \ldots, X_n) \mid \theta\} \quad = \quad 1 - \alpha,$$

(a one-sided lower $(1-\alpha)100\%$ random interval for θ), or

$$P\{\theta < g_2(X_1, \ldots, X_n) \mid \theta\} \quad = \quad 1 - \alpha,$$

(a one-sided upper $(1-\alpha)100\%$ random interval for θ). Derive and evaluate a 95% upper confidence interval for σ^2 using the dog data set of question 1.

5. Suppose we wish to find an interval estimator for the parameter p when we have a random variable $X \sim Bin(n, p)$.

(a) By considering the Normal approximation to the Binomial distribution, show that $\frac{X/n - p}{\sqrt{p(1-p)/n}}$ is an approximate pivot for p, and state its sampling distribution.

(b) Write down a random interval which contains p with probability (approximately) 0.95 (to do this you will need to make a further approximation of the variance of the Normal using a point estimator of p).