MA20226 - Question Sheet Three

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2011/12 Semester I

Set: Tutorials of Week 5, commencing Monday 31st October 2011.
Due in: Practicals of Week 6, commencing Monday 7th November 2011. You should write both your name and tutorial group on your work. If you are unable to make your practical then you should ensure that you hand the work to me personally *before* the practical is held; my office is 4W4.10.
Task: Attempt questions 1-3; questions 4-6 are additional questions and may be used for tutorial discussion.

1. Let $X_1, \ldots, X_n$ be independent and identically distributed random quantities, each having probability density function
$$f(x \,|\, \theta) = \begin{cases} \dfrac{x^3}{6\theta^4} \exp\left(-\dfrac{x}{\theta}\right) & 0 < x < \infty, \\ 0 & \text{otherwise} \end{cases}$$
for $0 < \theta < \infty$.
   (a) Prove that, for each $i = 1, \ldots, n$, $E(X_i \,|\, \theta) = 4\theta$ and $Var(X_i \,|\, \theta) = 4\theta^2$.
   (b) Find the maximum likelihood estimator, $T = T(X_1, \ldots, X_n)$, of θ.
   (c) Find the bias and mean square error of T.
   (d) Explain whether or not T is a consistent estimator of θ.

2. Calculate the mean, median and once trimmed mean for the following data:
$$4.3,\ 14.2,\ 15.3,\ 16.4,\ 17.5,\ 17.5,\ 19.7,\ 21.9,\ 23.0,\ 27.4$$
Now suppose that the observation with value 4.3 is replaced by a new observation with value x, where x takes some value between 0 and 30. Derive expressions for the three estimates calculated above as a function of x, and comment on their robustness.

3. Show that $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ may be computed as
$$s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right).$$

4. Let $X_1, \ldots, X_n$ be independent and identically distributed random quantities, each having probability density function
$$f(x \,|\, \alpha) = \begin{cases} \exp\{-(x-\alpha)\} & \alpha \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$
so that, for each $i = 1, \ldots, n$, $E(X_i \,|\, \alpha) = \alpha + 1$ and $Var(X_i \,|\, \alpha) = 1$.

(a) Deduce that $T_1 = \overline{X} - 1$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, is an unbiased estimator of α and find its mean square error.

(b) The maximum likelihood estimator of α is $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Without further calculation, explain why $X_{(1)}$ is a biased estimator of α.

(c) By first showing that, for each $i = 1, \ldots, n$,

$$P(X_i > x \,|\, \alpha) \quad = \quad \begin{cases} \exp\{-(x-\alpha)\} & \alpha \leq x < \infty, \\ 1 & \text{otherwise,} \end{cases}$$

show that the distribution of $X_{(1)}$ has probability density function

$$f_{X_{(1)}}(x \,|\, \alpha) \quad = \quad \begin{cases} n\exp\{-n(x-\alpha)\} & \alpha \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Given that $E(X_{(1)} \,|\, \alpha) = \alpha + \frac{1}{n}$ and $Var(X_{(1)} \,|\, \alpha) = \frac{1}{n^2}$ find the mean square error of $X_{(1)}$.

(e) Justifying your answers, state whether or not T_1 and $X_{(1)}$ are consistent estimators of α.

(f) What is the efficiency of $X_{(1)}$ relative to T_1? Which estimator, and why, do you prefer?

5. Use your statistical tables to find the following quantities.

(a) The values which enclose the central 90% of a $N(0,1)$ distribution.

(b) The values which enclose the central 90% of a $N(10,5)$ distribution.

(c) The values which enclose the central 95% of a χ^2_{15} distribution.

6. Suppose $X_1, \ldots, X_n$ are iid $U(0,\theta)$ random quantities, and that we wish to find an interval estimator for θ.

(a) Recall the cumulative distribution function of $M = \max\{X_1, \ldots, X_n\}$ (which you found on question 2(c) on Question Sheet Two)

$$P(M \leq m \,|\, \theta) \quad = \quad \begin{cases} 0 & m < 0 \\ \left(\frac{m}{\theta}\right)^n & 0 < m \leq \theta \\ 1 & m > \theta. \end{cases}$$

By considering a particular linear transformation of M, find a pivot for θ.

(b) Now using the cumulative distribution function of the pivot you have derived, find a random interval which contains θ with probability 0.95.

(c) Does this interval contain the maximum likelihood estimator of θ?