MA20226 - Feedback on Question Sheet Five

Simon Shaw, s.shaw@bath.ac.uk
http://people.bath.ac.uk/masss/ma20226.html

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1. The test statistic is $\sum_{i=1}^n x_i^2$ (or, equivalently, $\frac{1}{n}\sum_{i=1}^n x_i^2$). This should not surprise you if you recall that the maximum likelihood estimate of σ^2 when μ is known is $\frac{1}{n}\sum_{i=1}^n (x_i-\mu)^2$ (see question 1.(b) of Question Sheet Two) and in this case we have $\mu = 0$.

   In general with $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$ then
   $$\frac{X_i - \mu}{\sigma} \sim N(0,1) \quad \text{and} \quad \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$
   Hence, using the definition of the χ^2-distribution,
   $$\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \mu)^2 \sim \chi_n^2 \quad \text{and} \quad \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi_1^2. \tag{1}$$
   So, under H_0, we have $\mu = 0$ and $\sigma^2 = \sigma_0^2$ so that $\frac{1}{\sigma_0^2}\sum_{i=1}^n X_i^2 \sim \chi_n^2$ when H_0 is true. The distributions in (1) effectively show why $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$ for we can write
   $$\begin{aligned}\frac{1}{\sigma^2}\sum_{i=1}^n (X_i-\mu)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^n \{(X_i - \overline{X}) + (\overline{X} - \mu)\}^2 \\ &= \frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \overline{X})^2 + \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)^2 \\ &= \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)^2.\end{aligned}$$
   Using moment generating functions, it's straightforward to show that if $W = U + V$ with U and V independent and $W \sim \chi_n^2$ and $V \sim \chi_1^2$ then $U \sim \chi_{n-1}^2$ which would complete the argument that $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$.

2. Most people essentially got this question correct.

3. Parts (a), (b) and (c) were done well. Part (d) was an explicit illustration that the test used in question 2. was uniformally most powerful so that, for any $\mu > 105$, the power at μ under the test in question 2. is greater than the corresponding power at μ under the test in question 3.