MA20226 - Feedback on Question Sheet Two

Simon Shaw, s.shaw@bath.ac.uk
http://people.bath.ac.uk/masss/ma20226.html

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1. Almost everyone showed that $\overline{X}$ was the maximum likelihood estimator for μ (when σ^2 is known) and most people derived that $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ was the maximum likelihood estimate of σ^2 (when μ is known) as this satisfied $l'(\sigma^2)=0$. However, a common mistake was to claim that the corresponding maximum likelihood estimator was $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ which is not true. The table below summaries the maximum likelihood estimators for the parameters μ, σ^2 when $X_1,\ldots,X_n$ are iid $N(\mu,\sigma^2)$.

Parameters	Maximum likelihood estimator	Biased or unbiased?
μ unknown σ^2 known	$\overline{X}$ for μ	unbiased
μ known σ^2 unknown	$\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ for σ^2	unbiased
μ unknown σ^2 unknown	$\overline{X}$ for μ $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ for σ^2	unbiased biased

   The essence of this question was that when μ is known, the maximum likelihood estimator of σ^2 depends upon μ. If μ is additionally unknown then the maximum likelihood approach essentially replaces μ in the estimator of σ^2 by its estimator $\overline{X}$. One effect of this is that the estimator of σ^2 is no longer unbiased.

2. In order to get the solution for this question, you have to take care with the sample space. If $X \sim U(0,\theta)$ then the pdf is zero outside of $(0,\theta)$. Consequently, the likelihood function is only non-zero when $\theta \geq x_i\ \forall i = 1,\ldots,n$. This is equivalent to $\theta \geq m$ where $m = \max\{x_1,\ldots,x_n\}$. This makes sense: if you are trying to estimate θ, the maximum possible value, then having observed $x_1,\ldots,x_n$ you know that θ can not be less than $\max\{x_1,\ldots,x_n\}$.

3. Overall, this question was well done and most people found that $\alpha = \frac{n-1}{n+1}$. The corresponding estimator was thus $\frac{1}{n+1}\sum_{i=1}^{n}(X_i-\overline{X})^2$. Putting together this question with question 1. we note that if we consider estimating σ^2 using the estimator $T(\delta) = \frac{1}{\delta}\sum_{i=1}^{n}(X_i-\overline{X})^2$ then:
   - Choosing $\delta = n$ gives $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$, the maximum likelihood estimator of σ^2 (assuming μ is also unknown).
   - Choosing $\delta = n-1$ gives $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$, the unbiased estimator of σ^2.
   - Choosing $\delta = n+1$ gives $\frac{1}{n+1}\sum_{i=1}^{n}(X_i-\overline{X})^2$ which has the smallest mean square error amongst estimators of the form $T(\delta)$.