Additional Handwritten Notes for Lecture Two.

$X \sim Bin(n, \theta)$ $\quad f_X(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

$Y \sim NBin(r, \theta)$ $\quad f_Y(y|\theta) = \binom{y-1}{r-1} \theta^r (1-\theta)^{y-r}$

Observe $x = r$ and $y = n$: two experiments see the same number of successes in the same number of trials

$$f_X(x|\theta) = c' \theta^x (1-\theta)^{n-x}$$
$$= c f_Y(y|\theta)$$

SLP implies make the same inference about θ in both cases.

Birnbaum's Theorem.

$$(WIP \wedge WCP) \Longleftrightarrow SLP$$

Proof.

Both $SLP \Rightarrow WIP$ and $SLP \Rightarrow WCP$ straightforward. We'll show is that $(WIP \wedge WCP) \Rightarrow SLP$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two experiments with

$$f_{X_1}(x_1|\theta) = c(x_1, x_2) f_{X_2}(x_2|\theta)$$

for $c = c(x_1, x_2) > 0$. c is known as x_1, x_2 have been observed.

Consider the mixture experiment where I perform $\mathcal{E}_1$ with probability $p_1 = \frac{1}{c+1}$

and $\mathcal{E}_2$ with probability $p_2 = \frac{c}{c+1}$

Then:

probability of choosing ξ_1

$$f^*((1, x_1) \mid \theta) = \frac{1}{c+1} f_{X_1}(x_1 \mid \theta)$$

Involve the same experiment ξ^*

probability of choosing ξ_2

$$= \frac{c}{c+1} f_{X_2}(x_2 \mid \theta) \qquad \left(\text{as } f_{X_1}(x_1 \mid \theta) = c f_{X_2}(x_2 \mid \theta)\right)$$

$$= f^*((2, x_2) \mid \theta)$$

By the WIP, $\quad Ev(\xi^*, (1, x_1)) = Ev(\xi^*, (2, x_2))$

$$\underset{\text{WCP}}{\|} \qquad\qquad \|$$

$$Ev(\xi_1, x_1) \qquad\qquad Ev(\xi_2, x_2)$$

Putting them together gives $Ev(\xi_1, x_1) = Ev(\xi_2, x_2)$ which is the SLP.

$$\begin{array}{ccccc} \text{SLP} & \longrightarrow & \text{SSP} & \longrightarrow & \text{WSP} \\ \updownarrow & & & & \\ (\text{WCP} \cap \text{WIP}) & & & & \end{array}$$

Example stopping rule.

Sequence $A_i \subset \mathcal{X}_1 \times \cdots \times \mathcal{X}_i$ and

$$p_j(x_1, \ldots, x_j) = \begin{cases} 0 & \text{if } (x_1, \ldots, x_j) \in A_j \quad \text{STOP} \\ 1 & \text{if } (x_1, \ldots, x_j) \notin A_j \quad \text{CONTINUE} \end{cases}$$