Notes to accompany Lecture Nine.
Proof of the Duality Theorem.

Family of confidence procedures: $\mathbb{P}(\theta \in C(X;\alpha) \mid \theta) \geq 1-\alpha$ for all $\theta \in \Theta$ and $\alpha \in [0,1]$.

Family of significance procedures: $\mathbb{P}(p(X;\theta_0) \leq u \mid \theta = \theta_0) \leq u$ for every $u \in [0,1]$ and for all $\theta_0 \in \Theta$

① Let p be a family of significance procedures and consider

$$C(x;\alpha) = \{\theta \in \Theta : p(x;\theta) > \alpha\}$$

For any $\theta \in \Theta$, $\mathbb{P}(\theta \in C(X;\alpha) \mid \theta) = \mathbb{P}(p(X;\theta) > \alpha \mid \theta)$
$= 1 - \mathbb{P}(p(X;\theta) \leq \alpha \mid \theta)$

Now as p is super-uniform, $\mathbb{P}(p(X;\theta) \leq \alpha \mid \theta) \leq \alpha$ and so $1 - \mathbb{P}(p(X;\theta) \leq \alpha \mid \theta) \geq 1-\alpha$. Thus, $C(X;\alpha)$ is a family of significance procedures.

This is a nesting family because if $\alpha' > \alpha$ and $\theta \in C(x;\alpha')$ then

$$p(x;\theta) > \alpha' > \alpha$$

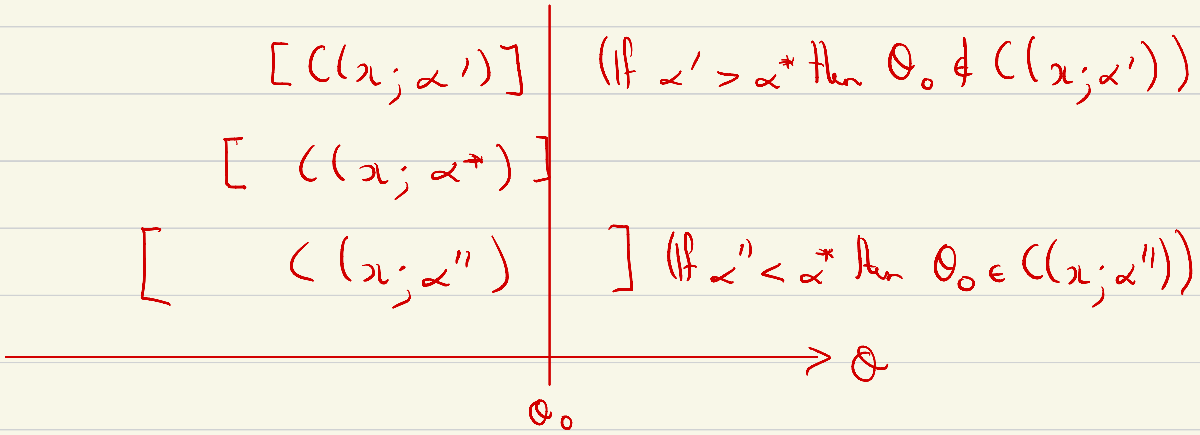
and so $\theta \in C(x;\alpha)$. Thus, $C(x;\alpha') \subset C(x;\alpha)$ and so C is nesting.

(2). Suppose that C is a nesting family. Thus, if $\theta_0 \notin C(x;\alpha)$ then $\theta_0 \notin C(x;\alpha')$ for $\alpha' > \alpha$.

Consider the smallest α for which $\theta_0 \notin C(x;\alpha)$

$$\alpha^* = \inf\{\alpha : \theta_0 \notin C(x;\alpha)\}$$

[For the boundary, assume that $\theta_0 \notin C(x;\alpha^*)$].



i.e. $\alpha \in [0, \alpha^*)$ then $\theta_0 \in C(x;\alpha)$
$\alpha \in [\alpha^*, 1]$ then $\theta_0 \notin C(x;\alpha)$

Thus, if $\theta_0 \notin C(x;u)$ then $\inf\{\alpha : \theta_0 \notin C(x;\alpha)\} \leq u$
and if $\inf\{\alpha : \theta_0 \notin C(x;\alpha)\} \leq u$ then $\theta_0 \notin C(x;u)$.

Hence, $\inf\{\alpha : \theta_0 \notin C(x;\alpha)\} \leq u \iff \theta_0 \notin C(x;u)$.

Let $p(x;\theta_0) = \inf\{\alpha : \theta_0 \notin C(x;\alpha)\}$ and take θ_0 and $u \in [0,1]$ to be arbitrary. We show that $p(X;\theta_0)$ is super-uniform.

$$P(p(X;\theta_0) \leq u \mid \theta_0) = \mathbb{P}(\theta_0 \notin C(X;u) \mid \theta_0)$$
$$\leq u$$ giving super-uniformity.

In both cases, if either procedure is exact then the inequalities may be replaced by equalities and so the dual procedure is also exact. $\square$

Probability Integral Transform.

If F is the distribution function of X then $\mathbb{P}(F(X) \leq x) \leq x$ i.e. it's super uniform.
If F is continuous, $\mathbb{P}(F(X) \leq x) = x$

$[\ \mathbb{P}(F(X) \leq x) = \mathbb{P}(X \leq F^{-1}(x)) = F(F^{-1}(x)) = x\ ]$.