Notes to accompany Lecture Four.

Suppose that $X = (X_1, \dots, X_n)$ and that the X_is are conditionally independent given θ with each $X_i \mid \theta \sim Po(\theta)$. Suppose that $\theta \sim Gamma(\alpha, \beta)$. Consider a point estimate of θ using the loss function $L(\theta, d) = (\theta - d)^2$.

- For the immediate problem, $[\Theta, \mathcal{D} = \Theta, f(\theta), L(\theta, d) = (\theta - d)^2]$

The Bayes rule is $d^* = \mathbb{E}(\theta) = \dfrac{\alpha}{\beta}$ with corresponding Bayes risk

$$\rho^*(f(\theta)) = Var(\theta) = \frac{\alpha}{\beta^2}.$$

- Solve for the decision after the sample, $[\Theta, \mathcal{D} = \Theta, f(\theta \mid x), L(\theta, d) = (\theta - d)^2]$.

$$
\begin{aligned}
f(\theta \mid x) &\propto f(x \mid \theta) f(\theta) \\
&\propto \left(\prod_{i=1}^{n} \theta^{x_i} e^{-\theta} \right) \theta^{\alpha - 1} e^{-\beta\theta} \\
&= \theta^{(\alpha + \sum_{i=1}^{n} x_i) - 1} e^{-(\beta + n)\theta}
\end{aligned}
$$

Hence, $\theta \mid x \sim Gamma(\alpha + \sum_{i=1}^{n} x_i, \beta + n)$

Bayes rule for the immediate decision

mle

The Bayes rule is $d^* = E(\theta \mid X) = \dfrac{\alpha + \sum_{i=1}^{n} x_i}{\beta + n} = \underbrace{\left(\dfrac{\beta}{\beta + n}\right)}_{k} \left(\dfrac{\alpha}{\beta}\right) + \underbrace{\left(\dfrac{n}{\beta + n}\right)}_{(1-k)} \left(\bar{x}\right)$

The Bayes rule is a weighted average of the Bayes risk for the immediate decision and the mle. Note that as n increases, the posterior decision is dominated by the data.

The corresponding Bayes risk is $Var(\theta \mid X) = \dfrac{\alpha + \sum_{i=1}^{n} x_i}{(\beta + n)^2}$. Note that this decreases in n.

In this case, $\delta^* = \dfrac{\alpha + \sum_{i=1}^{n} X_i}{\beta + n}$

The risk of the sampling procedure is

$$E[\mathrm{Var}(\Theta | X)] = E\left(\frac{\alpha + \sum_{i=1}^{n} X_i}{(\beta+n)^2} \right) = \frac{\alpha + \sum_{i=1}^{n} E(X_i)}{(\beta+n)^2}$$

Now, $E(X_i) = E(E(X_i | \Theta)) = E(\Theta) = \dfrac{\alpha}{\beta}$ as $\Theta \sim \mathrm{Gamma}(\alpha, \beta)$.

$X_i | \Theta \sim \mathrm{Po}(\Theta)$

Thus, $E[\mathrm{Var}(\Theta | X)] = \dfrac{\alpha + n\frac{\alpha}{\beta}}{(\beta+n)^2} = \dfrac{\alpha}{\beta(\beta+n)}$

Note that this decreases as n increases (and $n=0$ gives $\frac{\alpha}{\beta^2}$, the risk of the immediate decision)