Notes to accompany Lecture Five.

EXAMPLE.

$R(\theta, \delta) = E[\, L(\theta, \delta(X)) \mid \theta\,]$, expectation using $f_X(x \mid \theta)$

Use quadratic loss so $L(\theta, \delta(X)) = (\theta - \delta(X))^2$.

1). $\delta_1(x) = \bar{x}$ $\quad R(\theta, \delta_1) = E((\theta - \bar{X})^2 \mid \theta)$

As $\bar{X} \mid \theta \sim N(\theta, \sigma^2/n)$ then $R(\theta, \delta_1) = Var(\bar{X} \mid \theta) = \dfrac{\sigma^2}{n}$ Constant in θ.

2). $\delta_2(x) = \tilde{x}$, the median. Approximately, $\tilde{X} \mid \theta \sim N(\theta, \frac{\pi\sigma^2}{2n})$ so that

$$R(\theta, \delta_2) = E((\theta - \tilde{X})^2 \mid \theta)$$
$$= Var(\tilde{X} \mid \theta) = \frac{\pi\sigma^2}{2n}.$$
Constant in θ.

3) $\delta_3(x) = \mu_0$

$$R(\theta, \delta_3) = E((\theta - \mu_0)^2 \mid \theta) = (\theta - \mu_0)^2.$$
This depends upon θ.

4) $\delta_4(x) = \left(\dfrac{1}{\sigma_0^2} + \dfrac{n}{\sigma^2}\right)^{-1}\left(\dfrac{\mu_0}{\sigma_0^2} + \dfrac{n\bar{x}}{\sigma^2}\right)$

$$R(\theta, \delta_4) = E\left[\left\{\theta - \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{X}}{\sigma^2}\right)\right\}^2 \Bigg| \, \theta\right]$$

$$= \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-2} E\left[\left\{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)\theta - \left(\frac{1}{\sigma_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{X}\right)\right\}^2 \Bigg| \, \theta\right]$$

$$= \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-2} E\left[\left\{\frac{1}{\sigma_0^2}(\theta - \mu_0) + \frac{n}{\sigma^2}(\theta - \bar{X})\right\}^2 \Bigg| \, \theta\right]$$

[Post lecture note: I missed the power of -2 on the $\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)$ terms. I've corrected this now.]

As $E[\theta - \bar{X} \mid \theta] = 0$ then

$\mathrm{Var}(\bar{X} \mid \theta) = \frac{\sigma^2}{n}$

$$R(\theta, \delta_4) = \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-2}\left[\frac{1}{\sigma_0^4}(\theta - \mu_0)^2 + \frac{n^2}{\sigma^4}\overbrace{E((\theta - \bar{X})^2 \mid \theta)}^{}\right]$$

$$= \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-2}\left[\frac{1}{\sigma_0^2}\left(\frac{\theta - \mu_0}{\sigma_0}\right)^2 + \frac{n}{\sigma^2}\right]$$

$\theta \sim N(\mu_0, \sigma_0^2)$

This depends upon θ.

Proof.

$$E\{L(\theta, \delta(X))\} = \int_X \int_\theta L(\theta, \delta(x)) f(\theta, x)\, d\theta\, dx$$

Expectation over both θ and x

[In Lecture Four, for the Bayes rule theorem proof, we used $f(\theta, x) = f(\theta \mid x) f(x)$. In this proof, we want to utilise $f(x \mid \theta)$ and the prior for θ, $\pi(\theta)$].

$(f(\theta, x) = f_X(x \mid \theta)\pi(\theta))$

$$= \int_\theta \left\{ \int_X L(\theta, \delta(x)) f_X(x \mid \theta)\, dx \right\} \pi(\theta)\, d\theta$$

$$= \int_\theta R(\theta, \delta)\, \pi(\theta)\, d\theta.$$

Suppose δ^* is inadmissible and dominated by δ_1. Then is an open set C of θ when $R(\theta, \delta_1) < R(\theta, \delta^*)$ with $R(\theta, \delta_1) \leq R(\theta, \delta^*)$ elsewhere.

$$E\{L(\theta, \delta^*(X))\} = \int_{C^c} R(\theta, \delta^*)\pi(\theta)\, d\theta + \int_C R(\theta, \delta^*)\pi(\theta)\, d\theta$$

$$\geq \int_{C^c} R(\theta, \delta_1)\pi(\theta)\,d\theta + \int_C R(\theta, \delta^*)\pi(\theta)\,d\theta$$

[In C, $R(\theta, \delta^*) > R(\theta, \delta_1)$]

$$> \int_{C^c} R(\theta, \delta_1)\pi(\theta)\,d\theta + \int_C R(\theta, \delta_1)\pi(\theta)\,d\theta$$

$$= E\{L(\theta, \delta_1(X))\}$$

Hence, $E\{L(\theta, \delta^*(X))\} > E\{L(\theta, \delta_1(X))\}$ which is a contradiction to δ^* being the Bayes rule. $\square$