

**Travelling-waves for the FKPP equation  
via probabilistic arguments**

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**Abstract.** We outline a completely probabilistic study of travelling-wave solutions of the FKPP reaction-diffusion equation that are monotone and connect 0 to 1. The necessary asymptotics of such travelling-waves are proved using martingale and Brownian motion techniques. Recalling the connection between the FKPP equation and branching Brownian motion through the work of McKean and Neveu, we show how the necessary asymptotics and results about branching Brownian motion combine to give the existence and uniqueness of travelling waves of all speeds greater than or equal to the critical speed.

## 1. Introduction.

We consider the Fisher-Kolmogorov-Petrovski-Piscounov reaction-diffusion equation (FKPP)

$$(1.1) \quad \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + ru(u - 1)$$

where  $u = u(t, x)$  is a function on  $[0, \infty) \times \mathbb{R}$ . The FKPP equation has been much studied by both analytic techniques (see Fisher[**10**], Kolmogorov, Petrovskii and Piskunov[**13**]) and probabilistic methods (see Watanabe[**22**], McKean[**14**],[**15**], Bramson[**6**], Biggins[**3**],[**4**],[**5**], Friedlin[**11**], Neveu[**16**], Zhao & Elworthy [23], Elworthy, Truman & Zhao [9], to give just a few).

Particular attention has been paid to those solutions of the form  $u(t, x) = w(x - ct)$ , so that  $w$  satisfies

$$(1.2) \quad \frac{1}{2}w'' + cw' + rw(w - 1) = 0.$$

Such a solution  $w$  is known as a travelling-wave of speed  $c$ . Throughout this paper, we are concerned with *monotone travelling-waves of speed  $c$  which connect 0 to 1*, where  $w : \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable and is a travelling-wave of speed  $c$  satisfying (1.2) with  $w(x) \rightarrow 0$  as  $x \rightarrow -\infty$ ,  $w(x) \rightarrow 1$  as  $x \rightarrow \infty$  and  $w'(x) > 0$  for all  $x \in \mathbb{R}$ .

As we remind the reader in part three, but see McKean [14] and Neveu [16] in particular, if we consider a standard branching Brownian motion then the study of various martingales leads to *exact* formulas for the travelling-waves of speeds  $c > \sqrt{2r}$  and so gives *existence* results. A further argument reveals there are no monotone travelling-waves for speeds  $c < \sqrt{2r}$ . In the final section, some special treatment is required at the critical speed,  $c = \sqrt{2r}$ , and we outline a different probabilistic proof for the existence of travelling-waves.

However, the main results presented in this paper are concerned with the necessary asymptotic behaviour of travelling-waves. We give a probabilistic argument using a ‘one-particle’ Brownian motion which shows that given a travelling-wave of speed  $c \geq \sqrt{2r}$  its asymptotic behaviour must essentially look like the slowest decaying solution,  $f$ , to the linearization about the ‘target’ 1, so  $f$  satisfies

$$(1.3) \quad \frac{1}{2}f'' + cf' + rf = 0.$$

Combining this asymptotic result with the branching Brownian motion study clinches the *uniqueness* of the monotone travelling-waves from 0 to 1 modulo translations by a constant.

Although the asymptotic results are very well known, the method of proof is entirely probabilistic and quite a straightforward application of martingale and Brownian motion results. It is the intention to complete entirely probabilistic arguments for the study of travelling-waves for this and some related reaction-diffusion equations as previous studies have often alternated between analysis and probability to gain results. In particular, in the probabilistic section of Champneys et al. [7] we had to ‘resort’ to some analytic results about the necessary asymptotics of travelling-waves to gain uniqueness, yet now

the probabilistic argument used in this paper should adapt to cover such coupled reaction-diffusion equation cases (which correspond to multi-type branching Brownian motions). More importantly, we emphasise that there are situations (such as the continuous-type branching Brownian motion found in Harris and Williams [12] in which the type of each particle moves as an Ornstein-Uhlenbeck processes) where there currently are not analytic results of the desired asymptotics that will lead to uniqueness of travelling-waves. We hope that the ideas used in the main results of this paper may ultimately be extended to cover such awkward cases.

## 2. The asymptotics of travelling-waves of speeds $c \geq \sqrt{2r}$ .

We first consider travelling-waves with speeds that are strictly greater than the critical speed of  $\sqrt{2r}$ . Details of the martingale and Brownian motion results used in this section can be found in Rogers and Williams [17]&[18].

**THEOREM 2.1.** *Suppose  $w$  is a monotone travelling-wave of speed  $c > \sqrt{2r}$ . Then there exists an  $\tilde{x} \in \mathbb{R}$  such that*

$$1 - w(x) \sim e^{\lambda(x+\tilde{x})} \quad \text{as } x \rightarrow \infty,$$

where  $\lambda$  is the larger root of  $\frac{1}{2}\lambda^2 + c\lambda + r = 0$  (so that  $\lambda = -c + \sqrt{c^2 - 2r} < 0$ ).

*Proof.* Setting  $u = 1 - w$  we have  $u$  (strictly) decreasing from 1 to 0 satisfying

$$(2.1) \quad \frac{1}{2} u'' + cu' + ru(1 - u) = 0.$$

Then, if  $X$  is a Brownian motion under  $\mathbb{P}_0$ , the Feynman-Kac formula suggests that, under  $\mathbb{P}_0$ ,

$$u(X_t + ct) e^{r \int_0^t (1-u(X_s+cs)) ds}$$

is a positive martingale. This is indeed the case (Itô's formula verifies it is a local martingale and as  $0 \leq u \leq 1$  it is also bounded for each fixed time), so that

$$u(x) = \mathbb{E}_0^x \left\{ u(X_t + ct) e^{rt - r \int_0^t u(X_s+cs) ds} \right\}$$

and hence for  $\lambda \in \mathbb{R}$ ,

$$e^{-\lambda x} u(x) = \mathbb{E}_0^x \left\{ e^{-\lambda(X_t+ct)} u(X_t + ct) e^{-r \int_0^t u(X_s+cs) ds} \cdot e^{\lambda(X_t-x) + (r+c\lambda)t} \right\}.$$

Now if we choose  $\lambda$  such that  $r + c\lambda = -\frac{1}{2}\lambda^2$  the last term in the expectation above become a martingale which can change the drift of the Brownian motion. Thus, we take  $\lambda = -c + \sqrt{c^2 - 2r} < 0$ , so that (importantly)  $\lambda + c > 0$ , then with  $v(x) := e^{-\lambda x} u(x)$  and  $X$  a BM with drift  $\lambda$  under  $\mathbb{P}_\lambda$ , we have

$$v(x) = \mathbb{E}_\lambda^x \left\{ v(X_t + ct) e^{-r \int_0^t u(X_s+cs) ds} \right\}.$$

Whence

$$(2.2) \quad v(X_t + ct)e^{-r \int_0^t u(X_s + cs) ds}$$

is a  $\mathbb{P}_\lambda$ -martingale which is positive and so also almost surely convergent.

The first objective is to show that

$$\int_0^\infty u(X_s + cs) ds < +\infty \quad \text{a.s. under } \mathbb{P}_\lambda$$

so that we can essentially ‘get rid of the non-linear term’ and deduce that  $v(X_t + ct)$  must also converge under  $\mathbb{P}_\lambda$ . We first make the trivial note that a positive martingale either converges to zero or something strictly positive, then in either case taking logarithms of (2.2) and dividing by  $X_t + ct$  gives

$$\limsup_t \left\{ \frac{\ln v(X_t + ct)}{X_t + ct} - \frac{r}{X_t + ct} \int_0^t u(X_s + cs) ds \right\} \leq 0 \quad \text{a.s. under } \mathbb{P}_\lambda.$$

Since  $u(x) \rightarrow 0$  as  $x \rightarrow \infty$  and  $X_s + cs \rightarrow \infty$  as  $s \rightarrow \infty$  almost surely under  $\mathbb{P}_\lambda$  (recall,  $c + \lambda > 0$ ), we have  $u(X_s + cs) \rightarrow 0$  and so

$$\frac{1}{t} \int_0^t u(X_s + cs) ds \rightarrow 0 \quad \text{a.s. under } \mathbb{P}_\lambda.$$

Then as  $(X_t + ct)/t \rightarrow c + \lambda > 0$ , combining the last couple of results gives

$$\limsup_t \frac{\ln v(X_t + ct)}{X_t + ct} \leq 0 \quad \text{a.s. under } \mathbb{P}_\lambda$$

which on expanding back out in terms of  $u$  gives

$$\limsup_t \frac{\ln u(X_t + ct)}{X_t + ct} \leq \lambda.$$

We now deduce that  $u$  decays exponentially fast at least at rate  $\lambda < 0$  and this now guarantees that  $\int_0^t u(X_s + cs) ds < +\infty$  almost surely under  $\mathbb{P}_\lambda$ .

Then we now know that  $v(X_t + ct)$  is a positive, convergent  $\mathbb{P}_\lambda$ -submartingale. Further, since the tail  $\sigma$ -algebra of a drifting Brownian motion is trivial, we must have

$$v(X_t + ct) \rightarrow K \quad (\text{a.s. under } \mathbb{P}_\lambda)$$

for some constant  $K \geq 0$ , hence  $v(x) \rightarrow K$  as  $x \rightarrow \infty$ . Also,  $v$  is continuous and  $v(x) \rightarrow 0$  as  $x \rightarrow -\infty$  so that, in fact,  $v$  is *bounded* on  $\mathbb{R}$ . Hence,

$$v(x) = \mathbb{E}_\lambda^x \left\{ v(X_t + ct)e^{-r \int_0^t u(X_s + cs) ds} \right\} = K \mathbb{E}_\lambda^x \left\{ e^{-r \int_0^\infty u(X_s + cs) ds} \right\}$$

since we are dealing with a bounded positive martingale which must converge and be uniformly integrable, hence we also have  $K > 0$ .  $\square$

The change of measure used in the above proof is very significant in many of the probabilistic studies of the FKPP equation. In later sections we shall look at the connections with branching Brownian motion, where each travelling-wave of a speed  $c$  can be written as the Laplace transform of a martingale limit and this martingale in some sense measures the particles that have essentially behaved like Brownian motions with drift  $\lambda$  (that is, they are ‘near’ position  $\lambda t$  at time  $t$ ). Large deviation heuristics also link in here, see discussions in the related typed branching Brownian motion models of Champneys et al [7] and Harris & Williams [12]. The same change of measure is also used in Zhao & Elworthy [23] for the example in Theorem 4.1, which is a special case of the use of Hamilton Jacobi theory in KPP equations (see also Elworthy, Truman & Zhao [9]). Furthermore, the conditions ‘ $\alpha < \sqrt{2\hat{c}/D}$ ’ in Zhao & Elworthy [23], Theorem 4.1 which is from Friedlin’s conditions (N) should be compared with taking the largest root of  $1/2\lambda^2 + c\lambda + r$  in theorem 2.1 above.

We now consider travelling-waves with the critical speed of  $\sqrt{2r}$  which have slightly different behaviour and require a more delicate treatment in the proof of their asymptotic result.

**THEOREM 2.2.** *Suppose  $w$  is a monotone travelling-wave of speed  $\tilde{c} = \sqrt{2r}$ . Then there exists an  $\tilde{x} \in \mathbb{R}$  such that*

$$1 - w(x) \sim x e^{\tilde{\lambda}(x+\tilde{x})} \quad \text{as } x \rightarrow \infty,$$

where  $\tilde{\lambda}$  is the repeated root of  $\frac{1}{2}\lambda^2 + \tilde{c}\lambda + r = 0$  (so that  $\tilde{\lambda} = -\sqrt{2r}$ ).

*Proof.* The argument is similar to the non-critical cases previously dealt with, but slightly more care is needed and a different change of measure required.

The first problem occurs as  $1/2\lambda^2 + \sqrt{2r}\lambda + r = 1/2(\lambda + \sqrt{2r})^2$  gives us repeated roots at  $\lambda + c = 0$  (we crucially used  $\lambda + c > 0$  before). However, changing measure using  $\lambda = -c + \delta$  for *any* small  $\delta > 0$  and following the same ideas as before leads to

$$(2.3) \quad \limsup_x \frac{\ln u(x)}{x} \leq \lambda - \frac{\delta}{2} = -\sqrt{2r} + \frac{\delta}{2}.$$

Thus, we still obtain exponential decay in  $u$  at least as fast as the root we anticipated in the connected quadratic equation.

We next note that the linearization of the equation for  $u$  about 0 gives

$$\frac{1}{2}f'' + \sqrt{2r}f' + rf = 0$$

which now has solutions of the form  $e^{-\sqrt{2r}x}$  and  $xe^{-\sqrt{2r}x}$ . Considering writing  $u = fv$  where  $f$  is one of these solutions to the linearization gives us  $v$  satisfying

$$(2.4) \quad f \left( \frac{1}{2}v'' + \left\{ \sqrt{2r} + \frac{f'}{f} \right\} v' - ruv \right) = 0$$

and we now consider both cases.

Taking  $f(x) = e^{-\sqrt{2rx}}$  in (2.4) removes the ‘drift’ term in this equation completely which leaves  $v$  satisfying

$$\frac{1}{2}v'' - ruv = 0.$$

Hence, with the definition

$$v(x) := u(x)/e^{-\sqrt{2rx}}$$

is can easily be verified that

$$(2.5) \quad v(B_t)e^{-r \int_0^t u(B_s) ds}$$

is a positive local martingale when  $B$  is a Brownian motion. We use this fact later, but for now it is of limited use since we are interested in the asymptotic behaviour of  $u$  and  $B$  is recurrent.

However, taking  $f(x) = xe^{-\sqrt{2rx}}$  in (2.4) we must have that for  $x > 0$ ,  $v$  satisfies

$$\frac{1}{2}v'' + \frac{1}{x}v' - ruv = 0.$$

This is now more promising as we have a drift corresponding to a Bessel process (which does drift off to infinity) so we have a good chance of learning something about the tail behaviour of  $u$ . Then, this time using the definition

$$v(x) := u(x)/(xe^{-\sqrt{2rx}})$$

it is easily verified that

$$(2.6) \quad v(R_t)e^{-r \int_0^t u(R_s) ds}$$

is a positive local martingale when  $R$  is a Bessel process with associated generator  $\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x}$ . (Note that it is now suspected that we *only* have a local martingale since near zero  $v(x)$  behaves like  $1/x$ .) Positive local martingales are positive supermartingales and hence must converge.

Additionally,  $u$  decaying exponentially fast (see (2.3)) guarantees that

$$\int_0^\infty u(R_s) ds < +\infty \quad \text{a.s.}$$

( $R_t$  heads away from the origin ‘more-or-less centred close to  $\sqrt{t}$ ’, see Shiga and Watanabe [19] for details.)

Then,  $v(R_t)$  is convergent almost surely and as the tail  $\sigma$ -algebra of  $R$  is again trivial there exists a constant  $K \geq 0$  such that

$$v(R_t) \rightarrow K \quad \text{almost surely as } t \rightarrow \infty.$$

As  $R_t \rightarrow +\infty$  a.s., we deduce that

$$v(x) \rightarrow K \quad \text{as } x \rightarrow \infty.$$

Notice, as  $v$  is continuous on  $(0, \infty)$ , it is therefore *bounded* on any interval  $[\epsilon, \infty)$  for  $\epsilon > 0$ .

It remains for us to show the constant  $K$  is in fact strictly positive. We follow a path leading to contradiction: *suppose* that  $K = 0$ , then  $u(x)/(xe^{\tilde{\lambda}x}) \rightarrow 0$  as  $x \rightarrow \infty$ , whence the positive local martingale

$$\frac{u(R_t)}{R_t e^{\tilde{\lambda}R_t}} e^{-r \int_0^t u(R_s) ds} \rightarrow 0 \quad \text{a.s.}$$

Using the local martingale property, when  $x > y$  and  $T_y := \inf\{t > 0 : R_t = y\}$  we have

$$\begin{aligned} \frac{u(x)}{x e^{\tilde{\lambda}x}} &= \mathbb{E}^x \left\{ \frac{u(R_{t \wedge T_y})}{R_{t \wedge T_y} e^{\tilde{\lambda}R_{t \wedge T_y}}} e^{-r \int_0^{t \wedge T_y} u(R_s) ds} \right\} \\ &= \frac{u(y)}{y e^{\tilde{\lambda}y}} \mathbb{E}^x \left\{ e^{-r \int_0^{T_y} u(R_s) ds} ; T_y < \infty \right\} \end{aligned}$$

since the stopped local martingale is a uniformly integrable martingale (recall  $u(x)/(xe^{\tilde{\lambda}x})$  is bounded on  $[y, \infty)$ ).

Now, for the Bessel process we find that  $\mathbb{P}^x(T_y < \infty) = y/x$ , so

$$\frac{u(x)}{x e^{\tilde{\lambda}x}} \leq \frac{u(y)}{y e^{\tilde{\lambda}y}} \mathbb{P}^x(T_y < \infty) = \frac{u(y)}{x e^{\tilde{\lambda}y}}.$$

Then we have the positive continuous function  $u(x)/e^{\tilde{\lambda}x}$  decreasing in the region  $x > 0$  and clearly tending to zero as  $x \rightarrow -\infty$ , whence  $u(x)/e^{\tilde{\lambda}x}$  is then a bounded function on the real line.

Recall from (2.5) that  $\left\{ u(B_t)/e^{\tilde{\lambda}B_t} \right\} e^{-r \int_0^t u(B_s) ds}$  is a positive local martingale when  $B$  is a Brownian motion, so now we know  $u(B_t)/e^{\tilde{\lambda}B_t}$  is a positive submartingale that is bounded so it must converge. Since the tail  $\sigma$ -algebra of a Brownian motion is trivial, the convergence here must be to some constant. Then there exist a  $C \geq 0$  such that

$$u(B_t)/e^{\tilde{\lambda}B_t} \rightarrow C \quad \text{almost surely as } t \rightarrow \infty.$$

Yet  $B_t$  is recurrent with  $\limsup B_t = +\infty$  and  $\liminf B_t = -\infty$  so in fact  $u(x)/e^{\tilde{\lambda}x} \equiv C$ , but we already know  $\lim_{x \rightarrow -\infty} u(x)/e^{\tilde{\lambda}x} = 0$  so that  $C = 0$ .

Thus,  $u(x) \equiv 0$  and  $w = 1 - u$  is just the constant solution 1, *not* a travelling-wave from 0 to 1. This contradiction leads us back to having  $K > 0$ .  $\square$

### 3. Branching Brownian Motion and Martingales.

We consider a standard branching Brownian motion where each particle currently alive performs a standard Brownian motion and has single offspring particles at a constant rate  $r$  which move off independently from the birth position also like standard Brownian motions, also giving birth at a rate  $r$ , and so on. Once born particles live forever and they are labelled in order of birth, with  $N(t)$  representing the number of particles alive a time  $t$  and  $X_k(t)$  representing the position at time  $t$  of the  $k$ -th born particle for  $k = 1, \dots, N(t)$ .

We remind the reader that branching Brownian motion is closely connected to the reaction-diffusion equation (1.1), and in particular we recall some of the connections with travelling-waves solutions via martingales. See McKean [14] and Neveu [16] for more details of this, and see also the related Champneys et al. [7] and Biggins [5].

**3.1. Additive martingales.** It can be shown that

$$(3.1) \quad \sum_{k=1}^{N(t)} f(X_k(t) + ct)$$

is a local martingale if and only if  $f$  satisfies the linearized travelling-wave equation (1.3)

$$(3.2) \quad \frac{1}{2}f'' + cf' + rf = 0.$$

Thus, we find the parameterized family of ‘additive’ martingales

$$(3.3) \quad Z_\lambda(t) := \sum_{k=1}^{N(t)} e^{\lambda(X_k(t) + c_\lambda t)}$$

where  $c_\lambda := -r/\lambda - \frac{1}{2}\lambda$  and  $\lambda \in \mathbb{R}$ .

Analysis of these martingales reveals that

- if  $\lambda \leq -\sqrt{2r}$ ,  $Z_\lambda(t) \rightarrow 0$  almost surely,

where as

- if  $-\sqrt{2r} < \lambda \leq 0$ ,  $Z_\lambda(t) \rightarrow Z_\lambda(\infty)$  almost surely and in  $\mathcal{L}^1$  and

$Z_\lambda$  is a *uniformly integrable martingale with a strictly positive limit*.

See Neveu [16] for details, or the related Champneys et al. [7] for a multi-type Branching Brownian motion version.

**3.2. Multiplicative martingales and existence of monotone travelling-waves.** It is also true that

$$(3.4) \quad \prod_{k=1}^{N(t)} w(X_k(t) + ct)$$

is a local martingale if and only if  $w$  satisfies the travelling-wave equation (1.2)

$$(3.5) \quad \frac{1}{2}w'' + cw' + rw(w - 1) = 0.$$

We can define a ‘product’ martingale using the uniformly integrable  $Z_\lambda$  martingales (when  $-\sqrt{2r} < \lambda \leq 0$ ) by

$$M_\lambda(t) := \mathbb{E}(e^{-Z_\lambda(\infty)} | \mathcal{F}_t).$$

Then  $M_\lambda$  is a martingale bounded in  $[0, 1]$  and it is *uniformly integrable* with  $M_\lambda(t) \rightarrow M_\lambda(\infty) := e^{-Z_\lambda(\infty)}$  almost surely. Define

$$(3.6) \quad w_\lambda(x) := \mathbb{E}^x e^{-Z_\lambda(\infty)} = \mathbb{E}^0 e^{-e^{\lambda x} Z_\lambda(\infty)}$$

and observe that  $w_\lambda$  is monotone increasing and goes from 0 to 1. Noting that

$$(3.7) \quad Z_\lambda(\infty) = \sum_{k=1}^{N(t)} e^{\lambda(X_k(t) + c_\lambda t)} W_k$$

where the  $W_k$  for  $k = 1, \dots, N(t)$  are independent of one another and the process up to time  $t$ , each identically distributed like  $Z_\lambda(\infty)$  starting from one initial particle at the origin, we have

$$\begin{aligned} M_\lambda(t) &= \mathbb{E}(e^{-Z_\lambda(\infty)} | \mathcal{F}_t) = \mathbb{E} \left( \prod_{k=1}^{N(t)} e^{-e^{\lambda(X_k(t) + c_\lambda t)} W_k} | \mathcal{F}_t \right) \\ &= \prod_{k=1}^{N(t)} \mathbb{E} \left( e^{-e^{\lambda(X_k(t) + c_\lambda t)} W_k} | \mathcal{F}_t \right) \\ &= \prod_{k=1}^{N(t)} w_\lambda(X_k(t) + c_\lambda t). \end{aligned}$$

Hence, we have  $\prod_{k=1}^{N(t)} w_\lambda(X_k(t) + c_\lambda t)$  is a uniformly integrable martingale with

$$w_\lambda(x) = \mathbb{E}^x \prod_{k=1}^{N(t)} w_\lambda(X_k(t) + c_\lambda t)$$

and  $w_\lambda$  satisfies the travelling-wave equation

$$(3.8) \quad \frac{1}{2} w_\lambda'' + c_\lambda w_\lambda' + r w_\lambda (w_\lambda - 1) = 0.$$

This has given us the *existence* of monotone travelling-waves from 0 to 1 for all speeds  $c > \sqrt{2r}$ .

[We trivially note that, by symmetry, we also have monotone travelling-waves of speeds  $c < -\sqrt{2r}$  which decrease from 1 to 0.]

**3.3. Uniqueness of travelling-waves.** *If we have a monotone travelling wave,  $w$ , going from 0 to 1 of speed  $c > \sqrt{2r}$  then*

$$M(t) := \prod_{k=1}^{N(t)} w(X_k(t) + ct)$$

is a martingale bounded in  $[0, 1]$  so that, for all  $t \geq 0$ ,

$$w(x) = \mathbb{E}^x M(0) = \mathbb{E}^x M(t) = \mathbb{E}^x \prod_{k=1}^{N(t)} w(X_k(t) + ct).$$

Since we have a positive martingale it converges with  $M(t) \rightarrow M(\infty)$  almost surely and moreover, since  $M$  is bounded, it is also uniformly integrable with  $M(t) = \mathbb{E}(M(\infty)|\mathcal{F}_t)$  and  $w(x) = \mathbb{E}^x M(\infty)$ .

Define the ‘left-most’ particle’s position,

$$L(t) := \inf_{k \leq N(t)} X_k(t).$$

Recall, the critical martingale  $Z_{-\sqrt{2r}}(t) \rightarrow 0$  almost surely and  $c_{-\sqrt{2r}} = \sqrt{2r}$ , hence

$$0 \leq e^{-\sqrt{2r}(L_t + \sqrt{2r}t)} \leq Z_{-\sqrt{2r}}(t)$$

and we find

$$L(t) + \sqrt{2r}t \rightarrow +\infty \quad a.s.$$

The asymptotic result from earlier also tells us that there exists a constant  $\tilde{x} \in \mathbb{R}$  such that with  $\lambda = -c + \sqrt{c^2 - 2r}$ ,  $-\ln w(x) \sim 1 - w(x) \sim e^{\lambda(x + \tilde{x})}$  as  $x \rightarrow +\infty$ . Given any  $\epsilon > 0$ , there exists a  $D \in \mathbb{R}$  such that

$$1 - \epsilon \leq \frac{-\ln w(x)}{e^{\lambda(x + \tilde{x})}} \leq 1 + \epsilon, \quad \text{for all } x \geq D.$$

Hence, whenever  $L_t + ct \geq D$ ,

$$(1 - \epsilon)e^{\lambda\tilde{x}} Z_\lambda(t) \leq -\ln M(t) = -\sum_{k=1}^{N(t)} \ln w(X_k(t) + ct) \leq (1 + \epsilon)e^{\lambda\tilde{x}} Z_\lambda(t)$$

and on letting  $t \rightarrow \infty$ , since  $L_t + ct \rightarrow +\infty$  a.s., we have

$$(1 - \epsilon)e^{\lambda\tilde{x}} Z_\lambda(\infty) \leq -\ln M(\infty) \leq (1 + \epsilon)e^{\lambda\tilde{x}} Z_\lambda(\infty).$$

As  $\epsilon > 0$  was arbitrary, we have  $M(\infty) = e^{-e^{\lambda\tilde{x}} Z_\lambda(\infty)}$  almost surely. This gives

$$w(x) = \mathbb{E}^x M(\infty) = \mathbb{E}^{x + \tilde{x}} e^{-Z_\lambda(\infty)} =: w_\lambda(x + \tilde{x})$$

ans so  $w$  is in fact a *translate* of the travelling-wave  $w_\lambda$  from the last section satisfying (3.8). Monotone-travelling waves of speed  $c > \sqrt{2r}$  from 0 to 1 are therefore *unique* modulo translation by a constant.

Then we have used *only* probabilistic ideas and proofs to gain both *existence and uniqueness* of travelling waves of speeds  $c > \sqrt{2r}$ .

The important point to realise is that study of the ‘additive’ martingales is predominantly a ‘linearized’ problem and consequently we have their explicit form and they are by nature easier to study, however they control the limiting behaviour of many ‘product’ martingales and so they end up giving results about the much more difficult ‘non-linear’ problem.

**3.4. Non-existence of monotone travelling-waves of low speeds.** Suppose we have travelling-wave,  $w \in [0, 1]$ , going from 0 to 1 of speed  $c < \sqrt{2r}$ . Then, as in the previous section,

$$M(t) := \prod_{k=1}^{N(t)} w(X_k(t) + ct)$$

is a martingale bounded in  $[0, 1]$  so that, for all  $t \geq 0$ ,

$$w(x) = \mathbb{E}^x \prod_{k=1}^{N(t)} w(X_k(t) + ct)$$

and since we have a positive martingale it converges with  $M(t) \rightarrow M(\infty)$  almost surely and moreover, since  $M$  is bounded, it is also *uniformly integrable* with  $M(t) = \mathbb{E}(M(\infty)|\mathcal{F}_t)$  and, more importantly,

$$w(x) = \mathbb{E}^x M(\infty).$$

However, using the ‘left-most’ particle’s position,  $L(t)$ , and since  $w \in [0, 1]$ , we have

$$(3.9) \quad 0 \leq M(t) = \prod_{k=1}^{N(t)} w(X_k(t) + ct) \leq w(L(t) + ct).$$

The change of measure method found in Champneys et al. (1995) uses the convergence properties of the  $Z_\lambda$  martingales to show that

$$\frac{L(t)}{t} \rightarrow -\sqrt{2r} \quad \text{almost surely.}$$

Then as  $c < \sqrt{2r}$ ,  $L(t) + ct \rightarrow -\infty$  a.s. and since  $w(x) \rightarrow 0$  as  $x \rightarrow -\infty$ , we have from (3.9) that

$$0 \leq M(t) \leq w(L(t) + ct) \rightarrow 0 \quad a.s.$$

We have now identified the martingale limit  $M(\infty) = 0$  a.s., but now the uniform integrability implies that  $w(x) = \mathbb{E}^x M(\infty) = 0$  for all  $x \in \mathbb{R}$  and so  $w \equiv 0$  is *not* a travelling-wave going from 0 to 1, contradicting our initial supposition.

**3.5. Existence and uniqueness of travelling-waves at criticality.** At the critical wave speed  $\sqrt{2r}$  we need to concern ourselves with the martingales with parameter  $\tilde{\lambda} := -\sqrt{2r}$ , yet  $Z_{\tilde{\lambda}}(t) \rightarrow 0$  almost surely and so we cannot make a product martingale exactly as before. However, if we consider the ‘derivative in  $\lambda$ ’ martingales

$$Z'_{\lambda}(t) := \frac{\partial}{\partial \lambda} Z_{\lambda}(t) = \sum_{k=1}^{N(t)} (X_k(t) - E'_{\lambda} t) e^{\lambda X_k(t) - E_{\lambda} t}$$

we find that the critical one is

$$Z'_{\tilde{\lambda}}(t) = \sum_{k=1}^{N(t)} (X_k(t) + \sqrt{2rt}) e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})}$$

and it is this martingale that turns out to provide the link to the critical travelling-waves. We begin by considering the uniqueness of any critical speed travelling-waves, if they exist.

Suppose  $w$  is a monotone travelling-wave of speed  $\sqrt{2r}$  connecting 0 to 1. Let  $y \in \mathbb{R}$  be *fixed*, then

$$M^y(t) := \prod_{k=1}^{N(t)} w(y + X_k(t) + \sqrt{2rt})$$

is a martingale bounded in  $[0, 1]$ , so it is uniformly integrable with

$$M^y(t) \rightarrow M^y(\infty) \in [0, 1] \quad a.s.$$

Recalling the asymptotics from theorem 2.2, there exists  $\tilde{x} \in \mathbb{R}$  where for any  $\epsilon > 0$  and  $y \in \mathbb{R}$  there exists  $D \in \mathbb{R}$  such that

$$1 - \epsilon \leq \frac{x e^{\tilde{\lambda}(x+y+\tilde{x})}}{-\ln w(x+y)} \leq 1 + \epsilon \quad \forall x \geq D.$$

Hence, whenever  $L_t + \sqrt{2rt} \geq D$  we can sum this inequality over the particles alive at time  $t$  to give

$$(1 - \epsilon) \{-\ln M^y(t)\} \leq e^{\tilde{\lambda}(\tilde{x}+y)} Z'_{\tilde{\lambda}}(t) \leq (1 + \epsilon) \{-\ln M^y(t)\}.$$

Since  $L_t + \sqrt{2rt} \rightarrow \infty$  a.s., we get

$$\begin{aligned} \limsup_t e^{\lambda(\tilde{x}+y)} Z'_{\tilde{\lambda}}(t) &\leq (1 + \epsilon) \{-\ln M^y(t)\} \\ \liminf_t e^{\tilde{\lambda}(\tilde{x}+y)} Z'_{\tilde{\lambda}}(t) &\geq (1 - \epsilon) \{-\ln M^y(t)\} \end{aligned}$$

and since  $\epsilon > 0$  is arbitrary the critical ‘derivative’ martingale must converge with

$$Z'_{\tilde{\lambda}}(t) \rightarrow Z'_{\tilde{\lambda}}(\infty) \in [0, +\infty] \quad a.s.$$

and

$$M^y(t) \rightarrow M^y(\infty) = \exp -e^{\tilde{\lambda}(\tilde{x}+y)} Z'_{\tilde{\lambda}}(\infty) \quad a.s.$$

We readily see from the uniform integrability that

$$w(y) = \mathbb{E}^0 \exp -e^{\tilde{\lambda}(\tilde{x}+y)} Z'_{\tilde{\lambda}}(\infty)$$

Since, for any  $z \in \mathbb{R}$ , it is easy to see that  $\mathbb{E}^{x+z} M^y(\infty) = \mathbb{E}^x M^{y+z}(\infty)$ , we have

$$w_{\tilde{\lambda}}(x) := \mathbb{E}^x \exp -Z'_{\tilde{\lambda}}(\infty) = \mathbb{E}^0 \exp -e^{\tilde{\lambda}x} Z'_{\tilde{\lambda}}(\infty)$$

and so given any travelling-wave,  $w$ , of critical speed there exists a constant  $\tilde{x} \in \mathbb{R}$  such that  $w(y) = w_{\tilde{\lambda}}(\tilde{x} + y)$ . We have shown that travelling-waves of critical speed, *if they exist*, must be *unique* up to translation by a constant. Also note, using monotonicity and the fact that  $w$  connects 0 to 1, that for every  $z \in \mathbb{R}$ ,

$$\begin{aligned} 0 &= \lim_{y \rightarrow -\infty} w(y) = \mathbb{P}^z(Z'_{\tilde{\lambda}}(\infty) = 0), \\ 1 &= \lim_{y \rightarrow +\infty} w(y) = \mathbb{P}^z(Z'_{\tilde{\lambda}}(\infty) < +\infty), \end{aligned}$$

so, in fact,

$$Z'_{\tilde{\lambda}}(\infty) \in (0, +\infty) \quad a.s.$$

The existence of a critical speed travelling-wave connecting 0 to 1 also implies that the martingale  $Z'_{\tilde{\lambda}}$  converges to a strictly positive and finite limit.

We now offer a probabilistic proof for the *existence* of a critical travelling-wave. Neveu [16] considers the following construction of a Galton-Watson process. We run a branching Brownian motion starting with one particle at the origin but we freeze each particle when it first hits the space-time line given by  $\tau = x + \sqrt{2rt}$ . If we let  $K_\tau$  be the number of particles first crossing the line  $\tau = x + \sqrt{2rt}$ , it is easy to believe that we have a continuous-time Galton-Watson process (see Chauvin [8]). There is clearly no extinction and the process is supercritical, so  $\mathbb{P}(K_\tau = 0) = 0$  and  $\mathbb{P}(K_\tau \rightarrow \infty) = 1$ . Also, since  $L_t + \sqrt{2rt} \rightarrow \infty$  a.s. we have  $0 < K_\tau < \infty$  almost surely for all ‘times’  $\tau$ .

Consider the generating functions

$$f_\tau(\alpha) := \mathbb{E}(\alpha^{K_\tau}), \quad \alpha \in [0, 1], \tau \geq 0.$$

The branching property says that for  $\sigma, \tau \geq 0$

$$f_{\tau+\sigma}(\alpha) = f_\tau(f_\sigma(\alpha)).$$

The only fixed points of  $f_\tau$  are 0 and 1 (for  $\tau > 0$ ) and it is strictly increasing and continuous on  $[0, 1]$ . We can extend  $f$  by running backwards in time via the inverse functions,  $f_{-\tau} := (f_\tau)^{-1}$ .

Define a function  $\phi : \mathbb{R} \rightarrow (0, 1)$  by  $\phi(x) := f_{-x}(\alpha_0)$  for some fixed  $\alpha_0 \in (0, 1)$ . Then, we have  $\phi(-\infty) = 0$ ,  $\phi(+\infty) = 1$  and  $\phi'(x) > 0$ . In fact, it is this function  $\phi$  that

is a critical speed travelling-wave that we require. Neveu [16] uses David Williams' path decomposition of a drifting Brownian motion together with the fact that  $\phi' = -a \circ \phi$ , where  $a$  is the infinitesimal generator of the Galton-Watson process  $K$ , to show directly that  $\phi$  satisfies the critical travelling-wave equation. Alternatively, since  $f_\tau(\phi(\tau + \sigma)) = \phi(\sigma)$ , the branching property quickly yields that, for every  $x \in \mathbb{R}$ ,

$$M_x(\tau) := \phi(x + \tau)^{K_\tau}$$

is martingale. This martingale is bounded in  $[0, 1]$ , hence convergent and uniformly integrable with

$$M_x(\tau) \rightarrow M_x \quad \text{as } t \rightarrow \infty \quad \text{a.s.}$$

and  $\phi(x) = \mathbb{E}M_x(0) = \mathbb{E}M_x$ .

Given any  $t > 0$ , for  $\tau$  sufficiently large we have

$$K_\tau = \sum_{k=1}^{N(t)} K_{\tau - (X_k(t) + \sqrt{2rt})}^{(i)}$$

where the  $K^{(i)}$  are independent identically distributed like  $K$  and they are also independent of the branching Brownian motion up to time  $t$ . Then

$$\begin{aligned} M_x(\tau) &= \prod_{k=1}^{N(t)} \phi(x + \tau)^{K_{\tau - \{X_k(t) + \sqrt{2rt}\}}^{(i)}} \\ &= \prod_{k=1}^{N(t)} \phi\left(x + X_k(t) + \sqrt{2rt} + \tau - \{X_k(t) + \sqrt{2rt}\}\right)^{K_{\tau - \{X_k(t) + \sqrt{2rt}\}}^{(i)}} \\ &= \prod_{k=1}^{N(t)} M_{x + X_k(t) + \sqrt{2rt}}^{(i)}\left(\tau - (X_k(t) + \sqrt{2rt})\right) \end{aligned}$$

Letting  $\tau$  go to infinity gives us

$$M_x = \prod_{k=1}^{N(t)} M_{x + X_k(t) + \sqrt{2rt}}^{(i)}$$

and, finally, taking expectations and making use of the tower property reveals that

$$\phi(x) = \mathbb{E}^x \prod_{k=1}^{N(t)} \phi(X_k(t) + \sqrt{2rt})$$

and we have  $\phi$  as a travelling-wave of critical speed.

[Of course, the methods employed so far in this section (together with the asymptotic result of theorem 2.1) could be adapted to give existence and uniqueness for the travelling-waves of speeds greater than the critical one.]

On a final note, although we have already shown the existence and uniqueness of the critical travelling-wave, we would like to suggest an approach in the same spirit as the previous sections where we directly analyse the additive ‘derivative’ martingale and only then deduce the existence of a critical travelling-wave (rather than vice-versa as earlier in this section).

Starting with one particle at the origin, consider the martingale

$$U_t^x := \sum_{k=1}^{N(t)} (x + X_k(t) + \sqrt{2rt}) e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})} = xZ_{\tilde{\lambda}}(t) + Z'_{\tilde{\lambda}}(t).$$

Let

$$T_x := \inf \left\{ t \geq 0 : x + X_k(t) + \sqrt{2rt} \leq 0 \text{ for some } k \leq N(t) \right\}$$

and then, as  $T_x$  is a stopping time,  $U_{T_x \wedge t}^x$  is a martingale which is now positive and hence must converge to  $U_{T_x}^x \in [0, +\infty)$ . Since  $V_t := L_t + \sqrt{2rt} \rightarrow +\infty$  a.s. and  $V$  is continuous,

$$-\infty < V^* := \inf_{t \geq 0} V_t \leq 0 \quad a.s.$$

Then  $\mathbb{P}(T_x = +\infty) = \mathbb{P}(V^* > -x) \uparrow 1$  as  $x \rightarrow +\infty$ , but on the set  $\{T_x = +\infty\}$  we know that  $U_{t \wedge T_x}^x = xZ_{\tilde{\lambda}}(t) + Z'_{\tilde{\lambda}}(t)$  must converge and as  $Z_{\tilde{\lambda}}(t) \rightarrow 0$  this actually tells us that  $Z'_{\tilde{\lambda}}(t)$  must converge. Since  $\mathbb{P}(T_x = +\infty) \uparrow 1$ , we have gained the required convergence of the critical derivative martingale,

$$Z'_{\tilde{\lambda}}(t) \rightarrow Z'_{\tilde{\lambda}}(\infty) \in [0, \infty) \quad a.s.$$

Observe that for  $s, t \geq 0$  (and with the obvious notation)

$$\begin{aligned} Z'_{\tilde{\lambda}}(t+s) &= \sum_{k=1}^{N(t)} e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})} \sum_{j=1}^{N_k(s)} (X_{j,k}(s) + \sqrt{2r}(t+s)) e^{-\sqrt{2r}(X_{j,k}(s) - X_k(t) + \sqrt{2rs})} \\ &= \sum_{k=1}^{N(t)} e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})} \sum_{j=1}^{N_k(s)} (X_{j,k}(s) - X_k(t) + \sqrt{2rs}) e^{-\sqrt{2r}(X_{j,k}(s) - X_k(t) + \sqrt{2rs})} \\ &\quad + \sum_{k=1}^{N(t)} (X_k(t) + \sqrt{2rt}) e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})} \sum_{j=1}^{N_k(s)} e^{-\sqrt{2r}(X_{j,k}(s) - X_k(t) + \sqrt{2rs})} \end{aligned}$$

On letting  $s$  tend to infinity and remembering  $Z_{\tilde{\lambda}}(\infty) = 0$ , we see that

$$(3.10) \quad Z'_{\tilde{\lambda}}(\infty) = \sum_{k=1}^{N(t)} e^{-\sqrt{2r}(X_k(t) + \sqrt{2rt})} Z'_k$$

where  $Z'_k$  are independent of one another and of the process up to time  $t$ , each identically distributed like  $Z'_{\tilde{\lambda}}(\infty)$  started with one particle at the origin.

Notice, using the crucial decomposition (3.10),

$$\mathbb{P}(Z'_\lambda(\infty) = 0) = \mathbb{P}(Z'_k = 0, k = 1, \dots, N(t)) = \mathbb{E} \left\{ \mathbb{P}(Z'_\lambda(\infty) = 0)^{N(t)} \right\}$$

and so  $\mathbb{P}(Z'_\lambda(\infty) = 0)$  must be a solution of  $r(u^2 - u) = 0$ , the infinitesimal generator of the branching process  $N(t)$ , whence  $\mathbb{P}(Z'_\lambda(\infty) = 0) = 0$  or 1.

*Ideally*, in a future paper we hope to offer a direct proof that, in fact,  $Z'_\lambda(\infty) \in (0, +\infty)$  almost surely. This would enable us to build a monotone travelling-wave going from 0 to 1, exactly as done previously (see section (3.4)), with

$$w(x) := \mathbb{E}^0 \left( e^{-e^{\tilde{\lambda}x} Z'_\lambda(\infty)} \right) = \mathbb{E}^x \prod_{k=1}^{N(t)} w(X_k(t) + \sqrt{2rt}).$$

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