Branching Brownian motion with an inhomogeneous breeding potential

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Abstract

This article concerns branching Brownian motion (BBM) with dyadic branching at rate $\beta|y|^p$ for a particle with spatial position $y \in \mathbb{R}$, where $\beta > 0$. It is known that for $p > 2$ the number of particles blows up almost surely in finite time, while for $p = 2$ the expected number of particles alive blows up in finite time, although the number of particles alive remains finite almost surely, for all time. We define the right-most particle, R_t to be the supremum of the spatial positions of the particles alive at time t and study the asymptotics of R_t as $t \to \infty$. In the case of constant breeding at rate β the linear asymptotic for R_t is long established. Here, we find asymptotic results for R_t in the case $p \in (0, 2]$. In contrast to the linear asymptotic in standard BBM we find polynomial asymptotics of arbitrarily high order as $p \uparrow 2$, and a non-trivial limit for $\ln R_t$ when $p = 2$. Our proofs rest on the analysis of certain additive martingales, and related spine changes of measure.

1 Introduction and notation

We consider a branching Brownian motion with an inhomogeneous breeding potential. Each particle diffuses as a driftless Brownian motion and splits into two particles at rate $\beta|y|^p$, where $\beta > 0$, $p \in [0, 2]$ and $y \in \mathbb{R}$ is the particle's spatial position. Particles are immortal. The set of particles alive at time t is N_t, and then, for each $u \in N_t$, $Y_u(t)$ is the spatial position of particle u at time t. In the sequel we refer to this process as a $(\beta|y|^p; \mathbb{R})$-BBM, with probabilities $\{P^x : x \in \mathbb{R}\}$, where P^x is the law of the process started from a single particle at the point $x \in \mathbb{R}$. E^x will be the expectation operator for P^x.

It is known from Itô and McKean [8, pp 200–211] that quadratic breeding is a critical rate for population explosions. If the breeding rate were instead

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$\beta|y|^p$ for $p > 2$, the population would almost surely explode in a finite time. However, for the $(\beta y^2; \mathbb{R})$-BBM the *expected* number of particles blows up in a finite time, but the total number of particles alive remains finite almost surely, for all time. The fact that expectations for this process are not well behaved adds to the difficulty of its study.

We define $R_t := \sup_{u \in N_t} Y_u(t)$ to be the right-most particle in a $(\beta|y|^p; \mathbb{R})$-BBM. It is very well known that for $p = 0$, i.e., constant breeding at rate β, the right-most particle satisfies $\lim_{t\to\infty} t^{-1} R_t = \sqrt{2\beta}$, almost surely. More precise limit results concerning sub-linear correction terms in this asymptotic have also been found, see Bramson [2] and [3], for example.

In this paper we find the leading order term in the right-most particle asymptotic for all $p \in [0, 2]$; the method of proof in all cases is almost identical, but for ease of exposition we separate the case of the quadratic breeding potential.

Theorem 1. **a)** *$(\beta|y|^p; \mathbb{R})$-BBM, $p \in [0, 2)$.*

$$\lim_{t\to\infty} \frac{R_t}{t^{\hat{b}}} = \hat{a}$$

P^x-almost surely, where

$$\hat{a} := \Big(\frac{\beta}{2}(2-p)^2\Big)^{\frac{1}{2-p}} \quad \text{and} \quad \hat{b} := \frac{2}{2-p}.$$

b) *$(\beta y^2; \mathbb{R})$-BBM.*

$$\lim_{t\to\infty} \frac{\ln R_t}{t} = \sqrt{2\beta}$$

P^x-almost surely.

Thus we see polynomial spatial growth in the $(\beta|y|^p; \mathbb{R})$-BBM of arbitrarily high power as $p \uparrow 2$, and the right-most particle in the $(\beta y^2; \mathbb{R})$-BBM has spatial displacement that is, asymptotically, of exponential order; contrast this with the linear spread of standard BBM.

To prove Theorem 1 we use martingale arguments – in particular an additive martingale and a spine change of measure. Section 2 introduces the spine ideas and details the changes of measure we use, as well as defining the additive martingale which turns out to be the key tool in the proof of these results. In Section 3 we study the convergence properties of this martingale, and this allows us to prove the main results in Section 4.

2 Spines, measures and martingales

In this section we give the background results and notation required for the spine set-up. We give a more general formulation here than is required for the rest of the paper. Spine constructions were introduced by Chauvin and Rouault [4] in the context of standard branching Brownian motion, and since the series of papers Lyons et al. [12], Lyons [11] and Kurtz et al. [9] have found wide application in the theory of branching processes. For some more recent spine ideas applied to branching diffusions, see Kyprianou [10], or Engländer and Kyprianou [6], for example. For more details on the spine set-up we use below, see Hardy and Harris [7].

Recall that the $(\beta|y|^p;\mathbb{R})$-BBM, given by the point process $\mathbb{Y}_t := \{Y_u(t) : u \in N_t\}$, where N_t is the set of particles alive at time t, has associated probability measures P^x with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. All particles are labelled according to the Ulam-Harris convention so that, for example, 412 is 'the individual being the second child of the first child of the fourth child of the initial ancestor, $\varnothing$'. For two labels u, v the notation $v < u$ means that v is an ancestor of u, and $|u|$ denotes the generation of u.

A *spine*, ξ, is a distinguished infinite line of descent from the initial ancestor and is given by $\xi = \{\varnothing, \xi_1, \xi_2, \ldots\}$, where ξ_n is the label of the spine at the n-th generation and $u \in \xi$ means that either $u = \varnothing$ or $u = \xi_i$ for some $i \in \mathbb{N}$. It is natural to think of the spine as a single diffusing particle, and we denote its path by $\{\xi_t\}_{t\geq 0}$. Let $n = \{n_t\}_{t\geq 0}$ be the counting function for the number of fissions that have occurred along the path of the spine by time t, with the set of fission times themselves being $\{S_i\}_{i\in\mathbb{N}}$. In particular, ξ_{n_t} is the label of the spine at time t.

We now define several filtrations that will be very important in our later proofs. The augmented filtration $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$ for the $(\beta|y|^p;\mathbb{R})$-BBM with a distinguished spine is defined by $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, \{\xi_{n_s}\}_{s\leq t})$. Thus, in addition to the genealogy and paths of the particles alive at time t, the filtration $\tilde{\mathcal{F}}_t$ knows which particle is the spine at all times $s \in [0,t]$. Also define $\mathcal{G}_t := \sigma(\{\xi_s\}_{s\leq t})$ and $\tilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t, \{n_s, \xi_{n_s}\}_{s\leq t})$. This means that $\mathcal{G}_t$ knows only the spine's *motion*; besides this, $\tilde{\mathcal{G}}_t$ also knows the *spine's* genealogy and the *spine's* fission times; neither knows anything about what happens off the spine.

On the filtration $\tilde{\mathcal{F}}_t$ we will define a measure $\tilde{P}^x$ that is the law of the branching Brownian motion $\{\mathbb{Y}_t\}_{t\geq 0}$ with a distinguished spine. By this we mean that the $(\beta|y|^p;\mathbb{R})$-BBM may be constructed under $\tilde{P}$ as follows:

- the initial ancestor (the spine process, ξ_t) diffuses as a standard Brownian motion;
- the fission times on the spine occur as a Poisson process of instantaneous rate $\beta|\xi_t|^p$, which is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to continue the spine, and it repeats stochastically the behaviour of its parent;
- the other particle initiates, from its birth position, an independent $(\beta|y|^p;\mathbb{R})$-BBM with law $P^{\cdot}$.

(Note that $\tilde{P}^x$ is an extension of the original measure P^x, with $P = \tilde{P}|_{\mathcal{F}_\infty}$.)

With these augmented measures and filtrations we can define a one-particle martingale that we will use to change the behaviour of the spine. Let $g \in C^1(0,\infty)$ satisfy $\int_0^t g'(s)^2\,\mathrm{d}s < +\infty$ for all $t > 0$ and define

$$\tilde{M}_g(t) := e^{-\beta\int_0^t |\xi_s|^p\,\mathrm{d}s} 2^{n_t} \times \exp\bigg(\int_0^t g'(s)\,\mathrm{d}\xi_s - \int_0^t \frac{1}{2}g'(s)^2\,\mathrm{d}s\bigg).$$

Observe that $\tilde{M}_g$ is itself the product of two $\tilde{P}$-martingales, the first of which increases the breeding rate along the spine, and the second causes the spine to diffuse as an Brownian motion about the path g. More precisely, we define a measure $\tilde{\mathbb{Q}}_g$ via

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}_g^x}{\mathrm{d}\tilde{P}^x}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_g(t),$$

and then the $(\beta|y|^p;\mathbb{R})$-BBM with a distinguished spine can be re-constructed in law under $\tilde{\mathbb{Q}}_g^x$ as:

- the initial ancestor (the spine) diffuses as

  $$\mathrm{d}\xi_t = \mathrm{d}\tilde{B}_t + g'(t)\,\mathrm{d}t,$$

  where $\tilde{B}$ is a $\tilde{\mathbb{Q}}_g$-Brownian motion;
- the fission times on the spine occur as a Poisson process of instantaneous rate $2\beta|\xi_t|^p$, which is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to be the spine, and it repeats stochastically the behaviour of its parent;
- the other particle initiates, from its birth position, an independent $(\beta|y|^p;\mathbb{R})$-BBM with law $P^{\cdot}$.

Furthermore, our construction of the spine foundations in terms of filtrations and sub-filtrations allows us to define a measure $\mathbb{Q}_g := \tilde{\mathbb{Q}}_g|_{\mathcal{F}_\infty}$. It can be shown that

$$\frac{\mathrm{d}\mathbb{Q}_g^x}{\mathrm{d}P^x}\bigg|_{\mathcal{F}_t} = M_g(t), \tag{1}$$

where

$$M_g(t) = \sum_{u\in N_t} \exp\bigg(\int_0^t g'(s)\,\mathrm{d}Y_u(s) - \int_0^t \Big(\frac{1}{2}g'(s)^2 + \beta|Y_u(s)|^p\Big)\,\mathrm{d}s\bigg)$$

is an additive martingale for the $(\beta|y|^p;\mathbb{R})$-BBM. Since $\mathbb{Q}_g$ was defined as a restriction of $\tilde{\mathbb{Q}}_g$ we have that the path-wise constructions of $\mathbb{Y}$ under $\mathbb{Q}_g$ and $\tilde{\mathbb{Q}}_g$ coincide. It is the martingale M_g which turns out to be the key tool in proving Theorem 1.

3 Convergence properties of M_g

For notational convenience we have defined the martingale M_g for a general path g, although in this section we will consider only those classes of function g we need in order to prove Theorems 1.

Theorem 2. a) $(\beta|y|^p;\mathbb{R})$*-BBM,* $p \in [0,2)$. *Consider paths* $g(s) = as^b$ *for* $a > 0$ *and* $b > 1$.

(i) If either $b < \hat{b} = \frac{2}{2-p}$, *or* $b = \hat{b}$ *and* $a < \hat{a} = \left(\frac{\beta}{2}(2-p)^2\right)^{\frac{1}{2-p}}$, *then* M_g *is uniformly integrable and* $M_g(\infty) > 0$ *almost surely.*

(ii) If either $b > \hat{b}$, *or* $b = \hat{b}$ *and* $a > \hat{a}$, *then* $M_g(\infty) = 0$ *almost surely.*

b) $(\beta y^2;\mathbb{R})$*-BBM. Consider paths* $g(s) = e^{\lambda s}$, *for* $\lambda > 0$.

(i) If $0 < \lambda < \sqrt{2\beta}$, M_g *is uniformly integrable and* $M_\lambda(\infty) > 0$ *almost surely.*

(ii) If $\lambda > \sqrt{2\beta}$ *then* $M_g(\infty) = 0$ *almost surely.*

We do not treat the critical martingales, i.e., $a = \hat{a}$ and $b = \hat{b}$, or $\lambda = \sqrt{2\beta}$, as they are not needed to prove Theorem 1; we anticipate, however, that the martingale limits are 0 in these special cases. Essential to our argument is the spine change of measure and the following measure theoretic result, cf. Durrett [5, p. 242] or Athreya [1]. Let μ and ν be two measures on a probability space $(\Omega, \{\mathcal{F}\}_{t\ge 0}, \mathcal{F}_\infty)$, related by the Radon-Nikodým derivative

$$\left.\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right|_{\mathcal{F}_t} = Z(t),$$

for some ν-martingale Z. Define $\bar{Z} := \limsup_{t\to\infty} Z(t)$, so that $\bar{Z} = \lim_{t\to\infty} Z(t)$ almost surely under ν, and then

$$\bar{Z} < \infty \quad \mu\text{-a.s.} \quad \Longleftrightarrow \quad \nu(Z(\infty)) = Z(0) \tag{2}$$

$$\bar{Z} = \infty \quad \mu\text{-a.s.} \quad \Longleftrightarrow \quad \bar{Z} = 0 \quad \nu\text{-a.s.} \tag{3}$$

Related spine techniques for martingale convergence go back to Lyons et al. [12].

Proof of Theorem 2: $\mathcal{L}^1$ convergence parts. We will deal initially with the statements telling us when M_g is uniformly integrable (Theorem 2 parts a(i) and b(i)). The first step is to decompose the martingale M_g by conditioning on the spine's path, $\{\xi_t\}_{t\ge 0}$, and fission times, $\{S_u : u \in \xi\}$. By conditioning on $\tilde{\mathcal{G}}_\infty$ we have

$$\tilde{\mathbb{Q}}^x_g(M_g(t)|\tilde{\mathcal{G}}_\infty) = \sum_{u<\xi_t} \exp\left(\int_0^{S_u} g'(s)\mathrm{d}\xi_s - \int_0^{S_u}\left(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\right)\mathrm{d}s\right)$$
$$+ \exp\left(\int_0^t g'(s)\mathrm{d}\xi_s - \int_0^t\left(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\right)\mathrm{d}s\right),$$

and we refer to the two pieces of this decomposition as $\mathbf{sum}(t)$ and $\mathbf{spine}(t)$. We wish to show that the conditional expectation above is $\tilde{\mathbb{Q}}_g^x$-almost surely bounded as $t \to \infty$.

Under $\tilde{\mathbb{Q}}_g^x$ we have $\mathrm{d}\xi_s = \mathrm{d}\tilde{B}_s + g'(s)\,\mathrm{d}s$, where $\tilde{B}$ is a $\tilde{\mathbb{Q}}_g^x$-Brownian motion, and we obtain

$$\mathbf{spine}(t) = \exp\bigg(\int_0^t g'(s)\mathrm{d}\tilde{B}_s + \int_0^t \Big(\frac{1}{2}g'(s)^2 - \beta|\tilde{B}_s + g(s)|^p\Big)\,\mathrm{d}s\bigg).$$

For functions g of the forms given in the statement of Theorem 2 we have

$$\frac{\int_0^t g'(s)\,\mathrm{d}\tilde{B}_s}{\int_0^t g'(s)^2\,\mathrm{d}s} \to 0, \qquad \tilde{\mathbb{Q}}_g^x\text{-a.s.}$$

In particular, $\tilde{B}_s/s \to 0$ almost surely under $\tilde{\mathbb{Q}}_g^x$, and so for any $\varepsilon, \delta > 0$ there exist random times $T_\delta, T_\varepsilon < \infty$ such that

$$(1-\varepsilon)g(s)^p \le |\tilde{B}_s + g(s)|^p \le (1+\varepsilon)g(s)^p \quad \text{for all } s > T_\varepsilon, \tag{4}$$

and

$$-\delta\int_0^t g'(s)^2\,\mathrm{d}s \le \int_0^t g'(s)\,\mathrm{d}\tilde{B}_s \le \delta\int_0^t g'(s)^2\,\mathrm{d}s \quad \text{for all } t > T_\delta. \tag{5}$$

Hence, for some random, almost-surely finite, constant $C > 0$ we get the upper bound

$$\mathbf{spine}(t) \le C\exp\bigg(\int_{T_\delta \vee T_\varepsilon}^t \Big((1+\delta)\frac{1}{2}g'(s)^2 - \beta(1-\varepsilon)g(s)^p\Big)\,\mathrm{d}s\bigg) \tag{6}$$

$\tilde{\mathbb{Q}}_g^x$-almost surely for $t > T_\delta \vee T_\varepsilon$. We now consider the cases $p \in [0,2)$ and $p = 2$ separately.

a) Let $p \in (0,2)$, so that equation (6) gives

$$\mathbf{spine}(t) \le C\exp\bigg((1+\delta)\int_{T_\delta \vee T_\varepsilon}^t \Big(\frac{1}{2}(ab)^2 s^{2(b-1)} - \frac{1-\varepsilon}{1+\delta}\beta a^p s^{pb}\Big)\,\mathrm{d}s\bigg)$$

$\tilde{\mathbb{Q}}_g^x$-a.s. for $t > T_\delta \vee T_\varepsilon$. If $b < \hat{b} = \frac{2}{2-p}$ then the bound above is $\tilde{\mathbb{Q}}_g^x$-a.s. of exponential order $-ct^{pb+1}$ as $t \to \infty$, for some constant $c > 0$. If $b = \hat{b}$ then the second exponent in the bound above is

$$\Big(\frac{2a^2}{(2-p)^2} - a^p\beta\frac{1-\varepsilon}{1+\delta}\Big)\int_{T_\delta \vee T_\varepsilon}^t s^{\frac{2p}{2-p}}\,\mathrm{d}s, \tag{7}$$

and the sign of this expression is negative if and only if

$$a < \Big(\Big(\frac{1-\varepsilon}{1+\delta}\Big)\frac{\beta(2-p)^2}{2}\Big)^{\frac{1}{2-p}}.$$

Since δ, ε can be chosen arbitrarily small we have a decaying upper bound for $\mathbf{spine}(t)$ in the case $b = \hat{b}$ and $0 < a < \hat{a}$.

b) If $p = 2$ then we consider paths $g(s) = e^{\lambda s}$ and our bound from equation (6) is

$$\mathbf{spine}(t) \leq C \exp\left((1+\delta) \int_{T_\delta \vee T_\varepsilon}^{t} \Big(\frac{\lambda^2}{2} - \beta \frac{1-\varepsilon}{1+\delta} \Big) e^{2\lambda s} \, \mathrm{d}s \right)$$

$\tilde{\mathbb{Q}}_g^x$-a.s. for $t > T_\varepsilon$. We can choose δ, ε such that this decays for any $\lambda \in (0, \sqrt{2\beta})$.

Turning our attention to $\mathbf{sum}(t)$, we return to the general case $p \in [0, 2]$. From (6) we have, for $t > T_\delta \vee T_\varepsilon$,

$$\begin{aligned}\mathbf{sum}(t) \leq & \sum_{\substack{u < \xi_t \\ S_u \leq T_\delta \vee T_\varepsilon}} \exp\left(\int_0^{S_u} g'(s) \mathrm{d}\xi_s - \int_0^{S_u} \Big(\frac{1}{2} g'(s)^2 + \beta |\xi_s|^p \Big) \mathrm{d}s \right) \\ & + \sum_{\substack{u < \xi_t \\ S_u > T_\delta \vee T_\varepsilon}} C \exp\left(\int_{T_\delta \vee T_\varepsilon}^{S_u} \Big((1+\delta) \frac{1}{2} g'(s)^2 - \beta(1-\varepsilon) g(s)^p \Big) \mathrm{d}s \right).\end{aligned}$$

The first sum in the expression above is finite $\tilde{\mathbb{Q}}_g^x$-almost surely. To deal with the second sum, we note that, under $\tilde{\mathbb{Q}}_g^x$, the births along the spine are an inhomogeneous Poisson process with rate $2\beta|\xi_t|^p$ at time t, and so the total number of offspring to time t is Poisson distributed with expectation $2\beta \int_0^t |\xi_s|^p \, \mathrm{d}s$, conditional on $\mathcal{G}_\infty$. In view of equation (4) this is almost surely $o(\int_0^t (1+\varepsilon)|g(s)|^p \, \mathrm{d}s)$ as $t \to \infty$. However, as we saw above, the individual terms in the second sum decay like $\exp(-c \int_0^t |g(s)|^p \, \mathrm{d}s)$ for some constant $c > 0$.

Thus in all cases $p \in [0, 2]$, the decaying size of the summands dominates the growth in the number of summands and the Law of Large Numbers gives $\limsup_{t \to \infty} \tilde{\mathbb{Q}}_g^x(M_g(t) | \tilde{\mathcal{G}}_\infty) < +\infty$, $\tilde{\mathbb{Q}}_g^x$-almost surely.

Using Fatou's lemma we have

$$\begin{aligned}\tilde{\mathbb{Q}}_g^x(\liminf_{t \to \infty} M_g(t) | \tilde{\mathcal{G}}_\infty) &\leq \liminf_{t \to \infty} \tilde{\mathbb{Q}}_g^x(M_g(t) | \tilde{\mathcal{G}}_\infty) \\ &\leq \limsup_{t \to \infty} \tilde{\mathbb{Q}}_g^x(M_g(t) | \tilde{\mathcal{G}}_\infty) < +\infty,\end{aligned}$$

$\tilde{\mathbb{Q}}_g^x$-almost surely, which implies that $\liminf_{t \to \infty} M_g(t) < +\infty$, $\tilde{\mathbb{Q}}_g^x$-almost surely. Furthermore, $\liminf_{t \to \infty} M_g(t)$ is $\mathcal{F}_\infty$-measurable, and so, $\mathbb{Q}_g^x$-almost surely, we have $\liminf_{t \to \infty} M_g(t) < +\infty$.

In light of (1), $1/M_g(t)$ is a positive $\mathbb{Q}_g$-supermartingale, which converges almost surely, whence $M_g(t)$ converges $\mathbb{Q}_g$-almost surely. Combining these last few observations we have $\limsup_{t \to \infty} M_g(t) = \liminf_{t \to \infty} M_g(t) < +\infty$ $\mathbb{Q}_g$-almost surely. Using equation (2) with Scheffé's Lemma shows that M_g is $\mathcal{L}^1(P^x)$-convergent for the paths g required in Theorem 2. Additionally, positivity of M_g means that M_g is bounded in $\mathcal{L}^1(P^x)$ and hence M_g is uniformly integrable.

It remains to show that $P^x(M_g(\infty) = 0) = 0$. For this we use the following lemma, whose proof we give separately at the end of this section.

Lemma 3. *Let* $q : \mathbb{R} \to [0,1]$ *be such that* $M_t := \prod_{u\in N_t} q(Y_u(t))$ *is a* P-*martingale. Then* $q(x) \equiv q \in \{0,1\}$.

If $p(x) := P^x(M_g(\infty) = 0)$, then applying the branching Markov property we obtain, for $t > 0$,

$$p(x) = E^x\Big(P^x(M_g(\infty) = 0|\mathcal{F}_t)\Big) = E^x\bigg(\prod_{u\in N_t} p(Y_u(t))\bigg),$$

whence $\prod_{u\in N_t} p(Y_u(t))$ is a martingale. Since $E^x(M_g(\infty)) = M_g(0) > 0$ for the required paths g it follows from Lemma 3 that $p(x) \equiv 0$. ☐

Proof of Theorem 2: zero-limit parts. Since one of the particles alive at time t is the spine, we have that

$$M_g(t) \geq \exp\bigg(\int_0^t g'(s)\mathrm{d}\tilde{B}_s + \int_0^t \Big(\frac{1}{2}g'(s)^2 - \beta|\tilde{B}_s + g(s)|^p\Big)\,\mathrm{d}s\bigg).$$

We now restrict to our particular paths of interest again and use (4) and (5) to find a lower bound of

$$M_g(t) \geq C\exp\bigg(\int_{T_\delta\vee T_\varepsilon}^t \Big((1-\delta)\frac{1}{2}g'(s)^2 - \beta(1+\varepsilon)g(s)^p\Big)\,\mathrm{d}s\bigg) \tag{8}$$

$\tilde{\mathbb{Q}}_g^x$-almost surely for $t > T_\delta \vee T_\varepsilon$, where $C > 0$ is some random, almost-surely finite, constant.

A case analysis very similar to that done in the proof of $\mathcal{L}^1$ convergence above shows that this lower bound blows up when either: $p \in [0,2)$ and $b > \hat{b}$; $p \in [0,2)$, $b = \hat{b}$ and $a > \hat{a}$; or $p = 2$ and $\lambda > \sqrt{2\beta}$.

Consequently, in all these cases, $\limsup_{t\to\infty} M_g(t) = +\infty$, $\tilde{\mathbb{Q}}_g^x$-almost surely. This also holds $\mathbb{Q}_g^x$-almost surely, and recalling (3) we obtain the result. ☐

Proof of Lemma 3. Since M_t is a martingale we have

$$q(x) = E^x M_t = \tilde{E}^x M_t \leq \tilde{E}^x q(\xi_t),$$

and so $q(\xi_t)$ is a bounded submartingale, which must converge $\tilde{P}$-almost surely to some limit q_∞. But ξ_t is recurrent, so for $q(\xi_t)$ to converge it follows that that $q(x) \equiv q_\infty$ is constant. If $q_\infty \in [0,1)$ then $M_t \to 0$ because $N_t \to +\infty$, P^x-almost surely, and as M_t is uniformly integrable it must be the case that $q(x) = E^x M_\infty = 0$. So $q_\infty \in \{0,1\}$, as claimed. ☐

4 The right-most particle asymptotic

In this section we will prove results which, taken together, will yield Theorem 1.

Proposition 4. a) $(\beta|y|^p;\mathbb{R})$-*BBM*, $p \in [0,2)$. $\liminf_{t\to\infty} t^{-\hat{b}}R_t \geq \hat{a}$ P^x-*a.s.*
b) $(\beta y^2;\mathbb{R})$-*BBM*. $\liminf_{t\to\infty} t^{-1}\ln R_t \geq \sqrt{2\beta}$ P^x-*a.s.*

Proof of Proposition 4. a) Define

$$B_a := \Big\{\exists u : \liminf_{t\to\infty} t^{-\hat{b}} Y_u(t) = a\Big\}.$$

Then $B_a \in \mathcal{F}_\infty$ and $\mathbb{Q}_g^x(B_a) = \tilde{\mathbb{Q}}_g^x(B_a) = 1$, because under $\tilde{\mathbb{Q}}_g^x$ the spine is a Brownian motion following the path $g(s) = as^{\hat{b}}$. For $a \in (0, \hat{a})$, M_g is uniformly integrable by Theorem 2 and hence $P^x(M_g(\infty)) = 1$. By definition we have

$$P^x(\mathbf{1}_{B_a} M_g(\infty)) = \mathbb{Q}_g^x(B_a) = 1,$$

and because both $P^x(M_g(\infty) > 0) = 1$ and $P^x(M_g(\infty)) = 1$, it follows that $P^x(B_a) = 1$. This holds for all $a \in (0, \hat{a})$ and the result follows.
b) The proof of this part is almost identical: setting

$$B_\lambda := \Big\{\exists u : \liminf_{t\to\infty} t^{-1} \ln Y_u(t) = \lambda\Big\},$$

we find that $P^x(B_\lambda) = 1$ for all $\lambda \in (0, \sqrt{2\beta})$. □

To prove the required upper bounds we will need the following 0-1 laws.

Lemma 5. *For all $x, y \in \mathbb{R}$ and $a, b > 0$,*

$$P^x\Big(\limsup_{t\to\infty} t^{-1} \ln R_t > y\Big) \in \{0,1\},$$
$$P^x\Big(\limsup_{t\to\infty} t^{-b} R_t > a\Big) \in \{0,1\}.$$

Proof. Setting

$$q_1(x) = P^x\Big(\limsup_{t\to\infty} t^{-1} \ln R_t \le y\Big)$$
$$q_2(x) = P^x\Big(\limsup_{t\to\infty} t^{-b} R_t \le a\Big),$$

it is easy to check that $\prod_{u\in N_t} q_1(Y_u(t))$ and $\prod_{u\in N_t} q_2(Y_u(t))$ are martingales, and applying Lemma 3 yields the result. □

Proposition 6. a) *$(\beta|y|^p;\mathbb{R})$-BBM, $p \in [0,2)$. $\limsup_{t\to\infty} t^{-\hat{b}} R_t \le \hat{a}$ P^x-a.s.*
b) *$(\beta y^2;\mathbb{R})$-BBM. $\limsup_{t\to\infty} t^{-1} \ln R_t \le \sqrt{2\beta}$ P^x-a.s.*

Proof. From Lemma 5 we know that the events in Proposition 6 have probability 0 or 1. Our method of proof is to suppose that the limsups are too large almost surely and then obtain a contradiction by showing that this would cause the martingale M_f to fail to converge, for a suitably chosen function f.

We suppose that, for $p \in [0,2)$,

$$P^x\Big(\limsup_{t\to\infty} t^{-\hat{b}} R_t > a_0\Big) = 1 \tag{9}$$

for some $a_0 > \hat{a}$, and for $p = 2$ that

$$P^x\Big(\limsup_{t\to\infty} t^{-1}\ln R_t > \lambda_0\Big) = 1 \tag{10}$$

for some $\lambda_0 > \sqrt{2\beta}$.

Now choose $a_1 \in (\hat{a}, a_0)$ and set $f_1(t) := a_1 t^{\hat{b}}$, and also choose $\lambda_1 \in (\sqrt{2\beta}, \lambda_0)$ and set $f_2(t) := e^{\lambda_1 t}$. We will show that assumptions (9) and (10) force, respectively, the martingales M_{f_1} and M_{f_2} to fail to converge. (Note that if we can show that the probability in expression (9) is zero for any $a > a_0$, this will also imply that there the right-most particle cannot have a displacement that is asymptotically of order t^b for any $b > \hat{b}$.)

Let $D(f_1)$ be the space-time region bounded above by the curve $y = f_1(t)$ and below by the curve $y = -f_1(t)$, and define $D(f_2)$ analogously. We will use the notation f and $D(f)$ to talk about both of these functions or regions without mentioning the individual cases explicitly. The path of the initial particle, the spine ξ_t, is a $\tilde{P}^x$-Brownian motion and satisfies $|\xi_t|/t \to 0$ almost surely. Hence there exists almost surely a random time $T' < \infty$ such that $\xi_t \in D(f)$ for all $t > T'$.

Next we observe that the spine will, throughout its (infinite) lifetime, spend an infinite amount of time in any interval $[\varepsilon, 1)$ for $\varepsilon > 0$. During the time it spends in this interval it is giving birth to offspring at the points of an inhomogeneous Poisson process with rate bounded below by $\beta\varepsilon^p > 0$; this assures us of the existence of an infinite sequence, $\{T_n\}_{n\in\mathbb{N}}$, of birth times along the path of the spine when it is in the interval $[\varepsilon, 1)$, with $0 \le T' < T_1 < T_2 < \ldots$ and $T_n \to \infty$. Denote by u_n the label of the particle born at time T_n, for all $n \in \mathbb{N}$.

Each particle u_n gives rise to an independent $(\beta|y|^p;\mathbb{R})$-BBM with law $P^{\xi(T_n)}$. Almost surely, by hypotheses (9) and (10), each u_n has a descendant that leaves the space-time region $D(f)$. We wish to consider those particles u_n whose first descendant to leave the region $D(f)$ does so by crossing the positive bounding curve. Since the breeding potential is symmetric about the origin, and the particles u_n are born in the interval $[\varepsilon, 1)$, there is at least probability $\frac{1}{2}$ that the first descendant of each u_n to leave $D(f)$ crosses the positive bounding curve. Almost surely, therefore, an infinite number of the particles u_n have a descendant which does this, and we will label this infinite subsequence of the u_n by $\{v_n\}_{n\in\mathbb{N}}$. For an offspring u of the initial ancestor ξ we will use the notation S_u for its birth time.

Almost surely we have a sequence of times $\{J_n\}_{n\in\mathbb{N}}$ at which the descendants of the v_n first exit $D(f)$, i.e.

$$J_n := \inf\Big\{t > S_{v_n} : \exists w_n > v_n, Y_{w_n}(t) = f(t), Y_{w_n}(s) > -f(s)\forall s \in [T', t]\Big\},$$

and we have $J_n \to \infty$ as $n \to \infty$. To obtain a contradiction we consider the martingale M_f evaluated at the times J_n. Noting that $f'' > 0$, we have

$$\begin{aligned}
M_f(J_n) &\geq \exp\left(\int_0^t f'(s)\,\mathrm{d}Y_{w_n}(s) - \int_0^t \left(\frac{1}{2} f'(s)^2 + \beta |Y_{w_n}(s)|^p \right) \mathrm{d}s \right) \\
&= \exp\Bigg(f'(J_n) Y_{w_n}(J_n) - f'(0) Y_{w_n}(0) \\
&\qquad - \int_0^{J_n} \left(f''(s) Y_{w_n}(s) + \frac{1}{2} f'(s)^2 + \beta |Y_{w_n}(s)|^p \right) \mathrm{d}s \Bigg) \\
&\geq \exp\Bigg(f'(J_n) f(J_n) - f'(0)x - \int_0^{T'} \left(f''(s)\xi_s + \frac{1}{2} f'(s)^2 + \beta |\xi_s|^p \right) \mathrm{d}s \\
&\qquad - \int_{T'}^{J_n} \left(f''(s) f(s) + \frac{1}{2} f'(s)^2 + \beta |f(s)|^p \right) \mathrm{d}s \Bigg) \\
&\geq C \exp\left(f'(J_n) f(J_n) - \int_0^{J_n} \left(f''(s) f(s) + \frac{1}{2} f'(s)^2 + \beta |f(s)|^p \right) \mathrm{d}s \right) \\
&\geq C' \exp\left(\int_0^{J_n} \left(\frac{1}{2} f'(s)^2 - \beta |f(s)|^p \right) \mathrm{d}s \right),
\end{aligned}$$

for some random, almost surely finite constants C and C' which only depend on the path of ξ up to time T'. This lower bound for M_f increases unboundedly when we substitute in either of the functions f_1 or f_2, giving $\limsup_{t\to\infty} M_f(t) = +\infty$ almost surely, which is a contradiction.

Recalling Lemma 5 we must have that the probabilities in expressions (9) and (10) are both 0, and the results follow. □

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