HGS Heidelberg Compact Course

"Computational Methods for Uncertainty Quantification"

Exercise Sheet 1

1. (MC Methods – Slide 11) Confidence intervals for Monte Carlo estimates:

   (a) Using the Berry-Esseen bound derive a confidence interval for the estimate S_N/N and (upper and lower) bounds on the probability that μ falls into this confidence interval.

   (b) In the Buffon needle problem, we have
   $$\mathbf{E}\left[H_k\right] = p,\ \mathbf{Var}[H_k] = p(1-p),\ \mathbf{E}\left[|H_k - p|^3\right] = p(1-p)(1-2p+2p^2).$$
   Calculate the confidence interval for this problem in the case $N = 3408$, $\ell = 2.5$, $d = 3$, and thus check how likely it is that Lazzarini's machine would produce 1808 intersections and a relative accuracy of π of $8.5 \cdot 10^{-8}$.

2. (Predator-Prey Example – Slide 17) Show that the mean square error for the Monte Carlo estimator can be expanded as
   $$\mathbf{E}\left[\left(\mathbf{E}\left[Q\right] - \widehat{Q}_M\right)^2\right] = \left(\mathbf{E}\left[Q - Q_M\right]\right)^2 + \frac{\mathbf{Var}[Q_M]}{N}$$
   Hint: Note that $\mathbf{E}\left[Q\right]$ is constant and only $\widehat{Q}_M$ is actually random.

3. (Predator-Prey Example – Slide 23) Implementing standard Monte Carlo and antithetic sampling for the predator-prey example:

   (a) Find an estimate for $\mathbf{Var}\left[\frac{1}{2}(\widehat{Q}_{M,N} + \widehat{\tilde{Q}}_{M,N})\right]$ based on the sample variances and covariances of $\{Q_M^{(k)}\}$ and $\{\tilde{Q}_M^{(k)}\}$ defined above.

   (b) Implement the Monte Carlo method for the predator-prey system with $\overline{\mathbf{u}}_0 = [0.5, 2]^{\mathrm{T}}$, $\epsilon = 0.2$, $T = 6$, using explicit Euler discretisation, i.e.
   $$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) \ \text{ and } \ \mathbf{u}(0) = \mathbf{u}_0 \quad \longrightarrow \quad \mathbf{u}_{j+1} = \mathbf{u}_j + \Delta t\, \mathbf{f}(\mathbf{u}_j).$$
   Study the discretisation and MC errors and compute confidence intervals.

   (c) Implement also the antithetic estimator and compare the variance of the two estimators. How much is the variance reduced? Does this reduction depend on the selected tolerance TOL?

4. (Multilevel MC – Slide 33) MLMC complexity analysis:

   (a) Solve the constrained minimisation problem on Slide 30 to find the otimal numbers of samples on each level. (*Hint:* Use a Lagrange multiplier approach to include the constraint and then consider the first-order optimality constraints to find the minimum.)

   (b) Proof the complexity theorem.

5. (Multilevel MC – Slide 36) Implement the MLMC method for the predator-prey problem:

   (a) Implement the multilevel MC method for the predator-prey problem. Choose M_0 not too small to avoid stability problems with the explicit Euler method. Compare the cost to achieve a certain tolerance TOL for the mean square error (in terms of floating point operations) against your other two implementations (standard and antithetic MC estimator). How big is the computational gain?

   (b) Recall that $\alpha = \gamma = 1$ in that case. Verify this with your code. Compute $\mathbf{Var}[\widehat{Y}_\ell]$ and $\mathbf{Var}[\widehat{Q}_{M_\ell}]$ for a range of values of ℓ and M_0. What is the numerically observed rate β? Prove this theoretically.

   (c) Can you think of any further enhancements of your code?

P.T.O.

6. (Open ended question – Slide 37)

   (a) Think of a UQ question in your field of research and try to formulate a simple model problem that encapsulates the essential question. What type of uncertainty is it? How could you model it within your problem? Can you formulate a Monte Carlo simulation to estimate the uncertainties in a derived quantity of interest from your model? Are any of the variance reduction techniques we discussed applicable? Is there a natural model hierarchy that could be exploited in a multilevel algorithm?

   (b) Implement a simple Monte Carlo code to quantify the uncertainties. If your problem has natural model hierarchies and allows to couple them, try to estimate $\mathbf{Var}[\widehat{Y}_\ell]$ and $\mathbf{Var}[\widehat{Q}_{M_\ell}]$ in the same way as we did above to check whether multilevel Monte Carlo would be beneficial.

   (c) Implement a multilevel MC method for your problem. Do you achieve the gains that were predicted in (b)?