Worksheet Graduate Course “Brownian motion”

Please submit your answer to exactly three of the following exercises!

In all exercises \((B_t : t \geq 0)\) denotes a standard Brownian motion, and unexplained notation is taken from the lectures and lecture notes.

**Exercise 1.** Prove the following lower bound for Lévy’s modulus of continuity:

For every constant \(c < \sqrt{2}\), almost surely, for every \(\varepsilon > 0\) there exist \(0 < h < \varepsilon\) and \(t \in [0, 1 - h]\) with \(|B_{t+h} - B_t| \geq c\sqrt{h\log(1/h)}\).

**Exercise 2.** Show that, almost surely, there exists a time \(t\) at which \(D^*B(t) = 0\).

**Exercise 3.** Show that the graph of Brownian motion \([\{(t, B_t) : t \geq 0\} \subset \mathbb{R}^2\]

is a set of Hausdorff dimension \(\frac{3}{2}\) almost surely.

**Exercise 4.** Construct a stopping time \(T\) with respect to \((\mathcal{F}^+(t) : t \geq 0)\) such that \(\mathbb{E}T < \infty\) and \(B(T)\) is uniformly distributed on \([-1, 1]\).

*Hint:* You may use the Azéma-Yor embedding theorem but only for distributions taking *finitely many* values.

**Exercise 5.** Let \((S_j : j = 0, 1, \ldots)\) be a simple symmetric random walk on the integers started at \(S_0 = 0\). Show that, for every \(a \geq 0\),

\[
\lim_{n \to \infty} \mathbb{P}\{|S_j| \neq n\ \text{for all integers } 1 \leq j \leq an^2\} = \mathbb{P}\{\max_{0 \leq t \leq a} |B_t| < 1\}.
\]

**Exercise 6.** Show that the maximum process \(\{M_t : t \geq 0\}\) and the local time at zero \(\{L_t : t \geq 0\}\) have the same law.

*Hint:* Show that the maximum process \(\{M_t : t \geq 0\}\) can be recovered from \(\{M_t - B_t : t \geq 0\}\) by counting downcrossings and apply Lévy’s theorem.