



*Déviations pour les temps locaux  
d'auto-intersections*

RENCONTRE ORGANISÉE PAR :  
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06-10 décembre 2010

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Vol. 2, n° 1 (2010), p. 27-29.

[http://acirm.cedram.org/item?id=ACIRM\\_2010\\_\\_2\\_1\\_27\\_0](http://acirm.cedram.org/item?id=ACIRM_2010__2_1_27_0)

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*Texte mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# Upper tail asymptotics for the intersection local times of random walks in high dimensions

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## Abstract

In high dimensions two independent simple random walks have only a finite number of intersections. I describe the main result obtained in a joint paper with Xia Chen in which we determine the exact upper tail behaviour of the intersection local time.

Suppose that  $(X^{(1)}(n) : n \in \mathbb{N}), \dots, (X^{(p)}(n) : n \in \mathbb{N})$  are  $p$  independent identically distributed random walks started in the origin and taking values in  $\mathbb{Z}^d$ . We shall always assume that the increments of these random walks are symmetric with finite variance. The *intersection local time* of the random walks is defined as

$$I := \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \mathbf{1}\{X^{(1)}(i_1) = \cdots = X^{(p)}(i_p)\}.$$

Dvoretzky and Erdős [2] showed that

$$\mathbb{P}\{I < \infty\} = \begin{cases} 1 & \text{if } p(d-2) > d, \\ 0 & \text{otherwise.} \end{cases}$$

From now on we assume that  $p(d-2) > d$ , i.e. we are in supercritical dimensions. Khanin et al. [3] have studied the upper tail behaviour of  $I$ . They find that, for  $a$  sufficiently large,

$$\exp\{-c_1 a^{\frac{1}{p}}\} \leq \mathbb{P}\{I > a\} \leq \exp\{-c_2 a^{\frac{1}{p}}\}.$$

Our main result refines these asymptotics and provides insight into the mechanism behind the large deviation.

Let  $\mathbb{G}$  be the Green function of the random walk, defined by

$$\mathbb{G}(x) := \sum_{n=1}^{\infty} \mathbb{P}\{X(n) = x\}.$$

Note that we are following the (slightly unusual) convention of not summing over the time  $n = 0$ , which has an influence on the value  $\mathbb{G}(0)$ . Let  $q > 1$  be the conjugate of  $p$ , defined by  $p^{-1} + q^{-1} = 1$ . For every nonnegative  $h \in L^q(\mathbb{Z}^d)$  a bounded, symmetric operator

$$\mathfrak{A}_h : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)$$

is defined by

$$\mathfrak{A}_h g(x) = \sqrt{e^{h(x)} - 1} \sum_{y \in \mathbb{Z}^d} \mathbb{G}(x-y) g(y) \sqrt{e^{h(y)} - 1}.$$

Our main result is formulated in terms of the spectral radius

$$\|\mathfrak{A}_h\| := \sup \{ \langle g, \mathfrak{A}_h g \rangle : \|g\|_2 = 1 \}$$

of the operator  $\mathfrak{A}_h$ .

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Text presented during the meeting “Excess Self-Intersections & Related Topics” organized by Amine Asselah. 06-10 décembre 2010, C.I.R.M. (Luminy).

**Theorem:** *The upper tail behaviour of the intersection local time  $I$  is given as*

$$\lim_{a \uparrow \infty} \frac{1}{a^{1/p}} \log \mathbb{P}\{I > a\} = -p \inf \{ \|h\|_q : h \geq 0 \text{ with } \|\mathfrak{A}_h\| \geq 1 \},$$

where the right hand side is negative and finite.

I briefly summarize some ideas behind the proof. An interesting aspect of this is that we work with an infinite time horizon in the entire proof and avoid the use of Donsker–Varadhan large deviation theory.

By a Tauberian theorem (see [4]), for any nonnegative  $X$ ,

$$\lim_{k \uparrow \infty} \frac{1}{k} \log E \left[ \frac{X^k}{(k!)^p} \right] = -\kappa \iff \lim_{a \uparrow \infty} \frac{1}{a^{1/p}} \log P\{X > a\} = -pe^{\kappa/p}.$$

Hence it suffices to study the asymptotics of high integer moments of  $I$ . There is a formula for these moments, which is unfortunately quite involved due to the discrete time setting. We have

$$I^k = \sum_{x_1, \dots, x_k \in \mathbb{Z}^d} \prod_{j=1}^p \sum_{i_1, \dots, i_k=1}^{\infty} \prod_{\ell=1}^k \mathbf{1}\{X^{(j)}(i_\ell) = x_\ell\}.$$

and hence

$$\begin{aligned} \mathbb{E}I^k &= \sum_{x_1, \dots, x_k \in \mathbb{Z}^d} \left[ \sum_{i_1, \dots, i_k} \mathbb{E} \prod_{\ell=1}^k \mathbf{1}\{X(i_\ell) = x_\ell\} \right]^p \\ &= \sum_{x_1, \dots, x_k \in \mathbb{Z}^d} \left[ \sum_{m=1}^k \sum_{\pi \in \mathcal{E}_m} \mathbf{1}\{(x_1, \dots, x_k) \in \mathcal{A}(\pi)\} \sum_{\substack{j_1, \dots, j_m \\ \text{distinct}}} \mathbb{E} \prod_{\ell=1}^m \mathbf{1}\{X(j_\ell) = x_{\pi_\ell}\} \right]^p \\ &= \sum_{x_1, \dots, x_k \in \mathbb{Z}^d} \left[ \sum_{m=1}^k \sum_{\pi \in \mathcal{E}_m} \mathbf{1}\{(x_1, \dots, x_k) \in \mathcal{A}(\pi)\} \sum_{\sigma \in \mathfrak{S}_m} \prod_{\ell=1}^m \mathbb{G}(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}) \right]^p, \end{aligned}$$

where  $\mathfrak{S}_m$  is the symmetric group in  $m$  elements,  $\mathcal{E}_m$  the set of partitions  $\{\pi_1, \dots, \pi_m\}$  of  $\{1, \dots, k\}$  into  $m$  nonempty sets, and  $\mathcal{A}(\pi)$  the set of tuples  $(x_1, \dots, x_k)$  which are constant on the partitions.

Let  $A \subset \mathbb{Z}^d$  be finite and  $G: \mathbb{Z}^d \rightarrow (0, \infty)$  symmetric and  $p$ -summable. We show by combinatorial means that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} \log \frac{1}{k!} \sum_{x_1, \dots, x_k \in A} \left[ \sum_{m=1}^k \sum_{\pi \in \mathcal{E}_m} \mathbf{1}\{(x_1, \dots, x_k) \in \mathcal{A}(\pi)\} \sum_{\sigma \in \mathfrak{S}_m} \prod_{\ell=1}^m G(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}) \right]^p \\ = -p \log \inf \{ \|h\|_q : h \geq 0 \text{ with } \|\mathfrak{A}_h^A\| \geq 1 \}, \end{aligned}$$

where the self-adjoint operator  $\mathfrak{A}_h^A: L^2(A) \rightarrow L^2(A)$  is defined by

$$\mathfrak{A}_h^A g(x) = \sqrt{e^{h(x)} - 1} \sum_{y \in A} G(x - y) \sqrt{e^{h(y)} - 1} g(y).$$

Applying this with the Green's function in place of  $G$  gives the lower bound. The extension of the upper bound from finite sets  $A$  to the entire lattice is nontrivial, because all shifts of  $A$  produce the same exponential decay of the upper tails of the intersection local times.

For the upper bound we need to project the full problem onto a finite domain by wrapping it around a torus. Let  $A = [-N, N]^d \cap \mathbb{Z}^d$  and write  $x = 2Nz + y$  with  $y \in A$ ,  $z \in \mathbb{Z}^d$ . Then

$$\begin{aligned} \sum_{x_1, \dots, x_k \in \mathbb{Z}^d} \left[ \sum_{m=1}^k \sum_{\pi \in \mathcal{E}_m} \mathbf{1}\{(x_1, \dots, x_k) \in \mathcal{A}(\pi)\} \sum_{\sigma \in \mathfrak{S}_m} \prod_{\ell=1}^m \mathbb{G}(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}) \right]^p \\ \leq \sum_{y_1, \dots, y_k \in A} \left[ \sum_{m=1}^k \sum_{\pi \in \mathcal{E}_m} \mathbf{1}\{(y_1, \dots, y_k) \in \mathcal{A}(\pi)\} \sum_{\sigma \in \mathfrak{S}_m} \prod_{\ell=1}^m G(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}) \right]^p, \end{aligned}$$

so that the problem retains the given form, but with a different kernel

$$G(y) = \left( \sum_{z \in \mathbb{Z}^d} G^p(2Nz + y) \right)^{1/p}.$$

We now apply the result for the finite domain, and then let the period  $2N$  of the torus go to infinity.

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