Large deviation theory and applications
Application II

The Theorem of Bahadur and Rao
and
Large Portfolio Losses

Martin Herdegen

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Abstract

The Theorem of Bahadur an Rao is a refinement of the classical Cramér's Theorem. Instead of just calculating the exponential rate of decay of large deviation events, it gives a precise estimate of the large deviation probabilities. We use this theorem as well as Cramér's Theorem to calculate the tail distributions of total financial losses of a large portfolio and compare the quality of both approximations.

1 The Theorem of Bahadur and Rao

Lemma 1 *Let X be a real valued non-degenerate random variable, $\Lambda(\lambda) := \log \mathbb{E}[\exp(\lambda X)]$ and $\Lambda^*(x) := \sup\{\lambda \in \mathbb{R} : \lambda x - \Lambda(\lambda)\}$. Define $D_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}$. Then Λ is real analytic in $Int\{D_\Lambda\}$ with*

$$\Lambda'(\lambda) = \mathbb{E}\big[X \exp\big(\lambda X - \Lambda(\lambda)\big)\big] \tag{1}$$

$$\Lambda''(\lambda) = \mathbb{E}\big[X^2 \exp\big(\lambda X - \Lambda(\lambda)\big)\big] - \mathbb{E}\big[X \exp\big(\lambda X - \Lambda(\lambda)\big)\big]^2 > 0 \tag{2}$$

$$\Lambda'(\lambda) = x \Rightarrow \Lambda^*(x) := \lambda x - \Lambda(\lambda) \tag{3}$$

If moreover X is positive and bounded, then $D_\Lambda = \mathbb{R}$, $\Lambda^*(x) = \sup\{\lambda \geq 0 : \lambda x - \Lambda(\lambda)\}$, $\Lambda'(\lambda)$ is strictly nondecreasing for $\lambda \geq 0$ and

$$\Lambda'(0) = \mathbb{E}[X] \tag{4}$$

$$\lim_{\lambda \to \infty} \Lambda'(\lambda) = \operatorname{ess\,sup} X \tag{5}$$

Theorem 1 (Theorem of Bahadur and Rao) *Let $X_1, \ldots, X_n$ be real valued non-degenerate i.i.d. random variables with non-lattice[1] law μ and $S_n := \frac{1}{n}\sum_{i=1}^n X_n$. Let $\zeta \in Int\{D_\Lambda\}$, $\zeta > 0$ and $z = \lambda'(\zeta)$. Then*

$$\lim_{n \to \infty} J_n(z)\mathbb{P}(S_n \geq z) = 1$$

where $J_n(z) = \zeta\sqrt{\Lambda''(\zeta)2\pi n}\exp(n\Lambda^*(z))$

[1] if X_1 has a lattice law with lattice constant d^{-1} and $\mathbb{P}(X_1 = z) > 0$, then the theorem holds with modified limit $\frac{\zeta d}{1-\exp(-\zeta d)}$; confer [2] page 110 et seqq.

Remark 1 *The condition $\zeta > 0$ implies that $z > \mathbb{E}[X_1] \in [-\infty; \infty)$.*

Corollary 1 (Cramér's Theorem) *Under the assumptions of Theorem 1,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq z) = -\Lambda^*(z)$$

Proof Define $a_n := J_n(z)\mathbb{P}(S_n \geq z)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq z) = \lim_{n\to\infty} \frac{1}{n}\big(\log a_n - \log J_n(z)\big) = -\Lambda^*(z)$$

Proof of Theorem 1

1. Normalisation: Define the probability measure $\tilde{\mu}$ by $d\tilde{\mu} = \exp\big(\zeta x - \Lambda(\zeta)\big)\, d\mu$ and let $Y_i := (X_i - z)/\sqrt{\Lambda''(\zeta)}$. Then Y_i are i.i.d random variables satisfying

$$\mathbb{E}_{\tilde{\mu}}[Y_i] = \frac{1}{\sqrt{\Lambda''(\zeta)}}\big(\underbrace{\mathbb{E}_{\mu}\big[X_i \exp\big(\zeta X_i - \Lambda(\zeta)\big)\big]}_{\Lambda'(\zeta)=z} - z\big) = 0$$

$$\mathbb{E}_{\tilde{\mu}}[Y_i^2] = \frac{1}{\Lambda''(\zeta)} \mathbb{E}_{\tilde{\mu}}[X_i^2 - 2z\underbrace{(X_i - z)}_{\mathbb{E}_{\tilde{\mu}}[\dots]=0} - z^2]$$

$$= \frac{1}{\Lambda''(\zeta)}\Big(\mathbb{E}_{\mu}\big[X_i^2 \exp\big(\zeta X_i - \Lambda(\zeta)\big)\big] - \underbrace{\mathbb{E}_{\mu}\big[X_i \exp\big(\zeta X_i - \Lambda(\zeta)\big)\big]^2}_{\Lambda'(\zeta)^2=z^2}\Big) \overset{(2)}{=} 1$$

$$\mathbb{E}_{\tilde{\mu}}[Y_i^3] =: m_3 < \infty \quad (\Lambda \text{ is real analytic at } \zeta)$$

2. Integration: Define $\psi_n := \zeta\sqrt{n\Lambda''(\zeta)}$ and $W_n = \frac{\sum_{i=1}^n Y_i}{\sqrt{n}}$. Moreover let F_n be the distribution function of W_n under[2] the law $\tilde{\mu}$. Using $S_n = z + \sqrt{\Lambda''(\zeta)/n}\, W_n$ we get:

$$\begin{aligned}
\mathbb{P}(S_n \geq z) &= \mathbb{E}_{\mu}[\mathbb{1}_{\{S_n \geq z\}}] = \mathbb{E}_{\tilde{\mu}}\big[\exp\big(-n(\zeta S_n - \Lambda(\zeta))\big)\mathbb{1}_{\{S_n \geq z\}}\big] \\
&= \exp\big(-n(\underbrace{\zeta z - \Lambda(\zeta)}_{\Lambda^*(z)})\big)\mathbb{E}_{\tilde{\mu}}\big[\exp(-\underbrace{n\zeta\sqrt{\Lambda''(\zeta)/n}}_{\psi_n}\, W_n)\mathbb{1}_{\{W_n \geq 0\}}\big] \\
&= \exp(-n\Lambda^*(z)) \int_0^\infty \exp(-\psi_n x)\, dF_n
\end{aligned}$$

Hence, by an integration by parts and substitution $t = \psi_n x$:

$$\begin{aligned}
J_n(z)\mathbb{P}(S_n \geq z) &= \sqrt{2\pi}\, \psi_n \int_0^\infty \exp(-\psi_n x)\, dF_n \\
&= \sqrt{2\pi}\, \psi_n \Big(\exp(-\psi_n x) F_n(x)\Big|_0^\infty + \int_0^\infty \psi_n \exp(-\psi_n x) F_n(x)\, dx\Big) \\
&= \sqrt{2\pi} \int_0^\infty \psi_n^2 \exp(-\psi_n x)(F_n(x) - F_n(0))\, dx \\
&= \sqrt{2\pi} \int_0^\infty \psi_n \exp(-t)(F_n(t/\psi_n) - F_n(0))\, dt
\end{aligned}$$

[2] More precisely: $F_n(x) = \mathbb{E}_{\tilde{\mu}}(\mathbb{1}_{\{W_n \leq x\}})$

3. Berry-Esseen expansion of F_n: The Berry-Esseen expansion[3] of F_n yields:

$$\lim_{n\to\infty} \left\{\sqrt{n} \sup_{x\,\in\,\mathbb{R}} \left|F_n(x) - \Phi(x) - \frac{m_3}{6\sqrt{n}}(1-x^2)\phi(x)\right|\right\} = 0 \tag{6}$$

Here Φ and ϕ denote the distribution and density function of a standard Normal. Noting that $\psi_n = \mathcal{O}(\sqrt{n})$ and $\Phi(t/\psi_n) = \Phi(0) + \frac{t}{\psi_n}\phi(0) + \mathcal{O}(\frac{t^2}{n})$ we get by using (6) and three times the dominated convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty} J_n(z)\mathbb{P}(S_n \geq z) &= \lim_{n\to\infty} \sqrt{2\pi} \int_0^\infty \psi_n \exp(-t)(\Phi(t/\psi_n) - \Phi(0))\, dt \quad + \\
&\quad \lim_{n\to\infty} \sqrt{2\pi} \int_0^\infty \psi_n \exp(-t) \frac{m_3}{6\sqrt{n}}[(1-(t/\psi_n)^2)\phi(t/\psi_n) - \phi(0)]\, dt \\
&= \lim_{n\to\infty} \sqrt{2\pi} \int_0^\infty t\exp(-t)\phi(0)\, dt \quad + \quad 0 \\
&= \sqrt{2\pi}\, \phi(0) = 1
\end{aligned}$$

q.e.d.

Now we want to extend Theorem 1 to the more general setup where the X_i are still independent, but not longer all identically distributed. We assume instead that the X_i can be split up into $K \in \mathbb{N}$ different classes of i.i.d. random variables, the proportion of the sizes of these classes being fixed:

- Let $q_1, \ldots, q_K \in (0\,;1)$ such that $\sum_{\alpha=1}^K q_\alpha = 1$. For $n \in \mathbb{N}$ define $Q_\alpha(n)$ to be the integer closest[4] to nq_α such that $\sum_{\alpha=1}^K Q_\alpha(n) = n$, implying in particular $\lim_{n\to\infty} \frac{Q_\alpha(n)}{n} = q_\alpha$. To simplify notations we will often suppress the dependence of $Q_\alpha(n)$ on n.
- For $n \in \mathbb{N}$ let X_α, $X_{\alpha,i}$, $\alpha \in \{1,...,K\}$, $i \in \{1,...,Q_\alpha\}$ be independent real valued (non-degenerate) random variables such that $X_{\alpha,i} \overset{d}{=} X_\alpha$ $\forall i \in \{1,...,Q_\alpha\}$ and X_α have the non-lattice law μ_α. Let $S_n := \frac{1}{n}\sum_{a=1}^K \sum_{i=1}^{Q_\alpha} X_{\alpha,i}$.
- Define $\Lambda_n(\lambda) := \frac{1}{n}\log\mathbb{E}[\exp(\lambda n S_n)]$ and $\Lambda_\alpha(\lambda) := \log\mathbb{E}[\exp(\lambda X_\alpha)]$, $\alpha \in \{1,...,K\}$. By independence, $\Lambda_n(\lambda) = \sum_{a=1}^K \frac{Q_\alpha}{n}\Lambda_\alpha(\lambda)$. Thus we may define $\Lambda(\lambda) := \lim_{n\to\infty}\Lambda_n(\lambda) = \sum_{a=1}^K q_\alpha\Lambda_\alpha(\lambda)$ and $\Lambda^*(x) := \sup\{\lambda \in \mathbb{R} : \lambda x - \Lambda(\lambda)\}$. Finally, let $D_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}$ and $D_{\Lambda_\alpha} := \{\lambda \in \mathbb{R} : \Lambda_\alpha(\lambda) < \infty\}$, $\alpha \in \{1,...,K\}$. Since $D_\Lambda \subset D_{\Lambda_\alpha}$ $\forall\alpha \in \{1,...,K\}$, by linearity, Lemma 1 holds mutatis mutandis for this more general setup, too.

Corollary 2 (Theorem of Bahadur and Rao - generalised version) *Let $\zeta \in Int\{D_\Lambda\}$, $\zeta > 0$ and $z = \lambda'(\zeta)$. Then*

$$\lim_{n\to\infty} J_n(z)\mathbb{P}(S_n \geq z) = 1$$

[3] see [3] page 512 et seqq.

[4] More precisely: define recursively $Q_1(n) := \lfloor nq_1\rfloor$; $Q_\alpha(n) := \lfloor n\sum_{i=1}^\alpha q_\alpha\rfloor - \sum_{i=1}^{\alpha-1} Q_i(n)$, $\alpha \in \{2,...,K-1\}$; $Q_K(n) = n - \sum_{i=1}^{K-1} Q_i(n)$

where $J_n(z) = c_n\zeta\sqrt{\Lambda''(\zeta)2\pi n}\exp(n\Lambda^*(z))$ *and* $c_n := \exp\big(n(\Lambda_n(\zeta) - \Lambda(\zeta))\big)$

Remark 2 The constant c_n oscillates around 1 on small scale since $\log(c_n) = n(\Lambda_n(\zeta)-\Lambda(\zeta)) = \sum_{i=1}^{K}(Q_\alpha(n)-nq_\alpha)\Lambda_\alpha(\lambda)$ is uniformly bounded and oscillates around 0.

Proof of Corollary 2

1. Normalisation: For $\alpha \in \{1,\dots,K\}$ define the probability measure $\tilde{\mu}_\alpha$ by $d\tilde{\mu}_\alpha = \exp(\zeta x - \Lambda_\alpha(\zeta))\,d\mu_\alpha$ and let $Y_{\alpha,i} := (X_{\alpha,i} - z)/\sqrt{\Lambda''(\zeta)}$, $i \in \{1,\dots,Q_\alpha\}$. Then $Y_{\alpha,i}$ are independent random variables satisfying

$$\begin{aligned} E_{\tilde{\mu}_\alpha}[Y_{\alpha,i}] &= 0 \\ E_{\tilde{\mu}_\alpha}[Y_{\alpha,i}^2] &= \frac{\Lambda''_\alpha(\zeta)}{\Lambda''(\zeta)} \\ E_{\tilde{\mu}_\alpha}[Y_{\alpha,i}^3] &=: m_3^\alpha < \infty \end{aligned}$$

Define $\mu := \bigotimes_{\alpha=1}^{K}\mu_\alpha^{\otimes Q_\alpha}$ and $\tilde{\mu} := \bigotimes_{\alpha=1}^{K}\tilde{\mu}_\alpha^{\otimes Q_\alpha}$.

2. Integration: Define $\psi_n := \zeta\sqrt{n\Lambda''(\zeta)}$ and $W_n = (\sum_{a=1}^{K}\sum_{i=1}^{Q_\alpha} Y_{\alpha,i})/(d_n\sqrt{n})$, where $d_n := \sqrt{\frac{\Lambda''_n(\zeta)}{\Lambda''(\zeta)}}$; note that $\lim_{n\to\infty} d_n = 1$. Moreover let F_n be the distribution function of W_n under the law $\tilde{\mu}$. Using $S_n = z + d_n\sqrt{\Lambda''(\zeta)/n}\,W_n$ we get:

$$\begin{aligned} \mathbb{P}(S_n \geq z) = \mathbb{E}_\mu[\mathbb{1}_{\{S_n\geq z\}}] &= \mathbb{E}_{\tilde{\mu}}[\exp(-n(\zeta S_n - \Lambda_n(\zeta))\mathbb{1}_{\{S_n \geq z\}}] \\ &= \exp(-n(\zeta z - \Lambda(\zeta)))\exp(n(\Lambda(\zeta) - \Lambda_n(\zeta)) \;\cdot \\ &\quad\; \mathbb{E}_{\tilde{\mu}}[\exp(-n\zeta d_n\sqrt{\Lambda''(\zeta)/n}\,W_n)\mathbb{1}_{\{W_n\geq 0\}}] \\ &= \frac{1}{c_n}\exp(-n\Lambda^*(z))\int_0^\infty \exp(-d_n\psi_n x)\,dF_n \end{aligned}$$

Hence, by an integration by parts and substitution $t = \psi_n x$:

$$\begin{aligned} J_n(z)\mathbb{P}(S_n \geq z) &= \sqrt{2\pi}\,\psi_n\int_0^\infty \exp(-d_n\psi_n x)\,dF_n \\ &= \sqrt{2\pi}\int_0^\infty d_n\psi_n\exp(-d_n t)(F_n(t/\psi_n) - F_n(0))\,\mathrm{d}t \end{aligned}$$

3. Berry-Esseen expansion of F_n: The Berry-Esseen expansion[5] of F_n yields in this context:

$$\lim_{n\to\infty}\Big\{\sqrt{n}\sup_{x\,\in\,\mathbb{R}}\Big|F_n(x) - \Phi(x) - \frac{m_{(3)}}{6d_n^3\sqrt{n}}(1-x^2)\phi(x)\Big|\Big\} = 0 \qquad (7)$$

where $m_{(3)} := \sum_{\alpha=1}^{K}\frac{Q_\alpha}{n}m_3^\alpha$. Note that $m_{(3)}$ is uniformly bounded. Recalling $\lim_{n\to\infty} d_n = 1$ we get by using (7) and four times the dominated convergence theorem:

$$\lim_{n\to\infty} J_n(z)\mathbb{P}(S_n \geq z) = \lim_{n\to\infty}\sqrt{2\pi}\int_0^\infty \psi_n\exp(-t)(F_n(t/\psi_n) - F_n(0))\,dt = 1$$

[5] see [3] page 521 et seqq.

Corollary 3 (Cramér's Theorem - generalised version) *Under the assumptions of Corollary 2,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq z) = -\Lambda^*(z)$$

Proof Define $a_n := J_n(z)\mathbb{P}(S_n \geq z)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq z) = \lim_{n\to\infty} \frac{1}{n}\big(\log a_n - \log J_n(z)\big) = -\Lambda^*(z)$$

2 The Portfolio Loss Model

2.1 The model

We consider (*bank*) portfolios consisting of positions of K different types (*rating classes*). We denote by q_α, $\alpha \in \{1, ..., K\}$, the proportion of positions of type α to the whole portfolio and regard these proportions to be fixed. For $n \in \mathbb{N}$ we define $Q_\alpha(n)$ to be the integer closest[6] to nq_α such that $\sum_{\alpha=1}^{K} Q_\alpha = n$. We are interested in the value of a portfolio of size n after one period (*e.g. a month*). More precisely, we are interested in the gross loss of the portfolio after one period. Therefore we work with the gross losses of each position rather than with the positions themselves. We define random variables Z_α, $Z_{\alpha,i}$, U_α, $U_{\alpha,i}$, $\alpha \in \{1, ..., K\}$, $i \in \{1, ..., Q_\alpha\}$ and Y with the following properties:

1. $Z_{\alpha,i} \overset{d}{=} Z_\alpha$, $U_{\alpha,i} \overset{d}{=} U_\alpha$, $\alpha \in \{1, ..., k\}$, $i \in \{1, ..., Q_\alpha\}$.
2. Y is a discretely valued "*macro-environmental*" random variable describing the state of the *business cycle* after one period.
3. $Z_\alpha \mid Y \sim Ber(p_\alpha(Y))$, $p_\alpha(Y) \in [0\,;1]$[7]; $p_\alpha(Y)$ is the probability of loss of a position of type α given Y.
4. There exists[8] $C > 0$ such that $0 < U_\alpha \leq C$ *a.s.*, $\alpha \in \{1, ..., K\}$; U_α describes the amount of money that would be lost in one position of type α in the event there is a loss (*total loss* $-$ *recovery rate*).
5. For all $\alpha \in \{1, ..., k\}$ and for all $i \in \{1, ..., Q_\alpha\}$ U_α and Y as well as $U_{\alpha,i}$ and Y are independent; i.e. the amount of loss of a certain position does not depend on the state of the business cycle.
6. **Conditional on Y**, Z_α, $Z_{\alpha,i}$, U_α and $U_{\alpha,i}$ are independent, $\alpha \in \{1, ..., k\}$, $i \in \{1, ..., Q_\alpha\}$.

The total loss of the portfolio is given by $L_n := \sum_{a=1}^{K} \sum_{i=1}^{Q_\alpha} Z_{\alpha,i} U_{\alpha,i}$. The main goal is now to calculate for given $x > 0$, as precisely as possible, the loss probability $\mathbb{P}(L_n \geq nx)$.

[6] confer above

[7] We assume $\forall y \, \exists \alpha \in \{1, ..., K\} : p_\alpha(y) > 0$.

[8] This condition is mathematically not really necessary, but economically trivial. It simplifies the argumentation since then all considered Laplace transforms exist.

2.2 Calculation of $\mathbb{P}(L_n \geq nx)$

For $\eta \in \mathbb{R}$ define $\Lambda_n(\eta|Y) := \frac{1}{n}\log \mathbb{E}[\exp(\eta L_n)|Y]$ and $\Lambda_\alpha(\eta|Y) := \log \mathbb{E}[\exp(\eta Z_\alpha U_\alpha)|Y]$, $\alpha \in \{1, ..., K\}$. Using the above properties we get:

$$\Lambda_\alpha(\eta|Y) = \log\left[(1 - p_\alpha(Y)) + p_\alpha(Y)\mathbb{E}[\exp(\eta U_\alpha)]\right]$$

$$\Lambda_n(\eta|Y) = \sum_{a=1}^{K} \frac{Q_\alpha(n)}{n}\Lambda_\alpha(\eta|Y)$$

$$\Lambda(\eta|Y) := \lim_{n\to\infty} \Lambda_n(\eta|Y) = \sum_{a=1}^{K} q_\alpha \Lambda_\alpha(\eta|Y)$$

To calculate $\mathbb{P}(L_n \geq nx)$ we use the law of total probability and get:

$$\mathbb{P}(L_n \geq nx) = \sum_y \mathbb{P}(Y = y)\mathbb{P}(L_n \geq nx|y)$$

Thus the problem boils down to question how to calculate $\mathbb{P}(L_n \geq nx|y)$ for a given value y. Let $\bar{x}(Y) := \lim_{n\to\infty} \mathbb{E}\left[\frac{L_n}{n}|Y\right]$ and $\hat{x}(Y) := \lim_{n\to\infty} \operatorname{ess\,sup} \mathbb{E}\left[\frac{L_n}{n}|Y\right]$. Using the above properties we get:

$$\begin{aligned}
\bar{x}(Y) &= \lim_{n\to\infty} \sum_{a=1}^{K} \frac{Q_\alpha(n)}{n}\mathbb{E}[Z_\alpha U_\alpha|Y] = \sum_{a=1}^{K} q_\alpha \mathbb{E}[Z_\alpha U_\alpha|Y] \\
&= \sum_{a=1}^{K} q_\alpha p_\alpha(Y)\mathbb{E}[U_\alpha] \\
\hat{x}(Y) &= \lim_{n\to\infty} \sum_{a=1}^{K} \frac{Q_\alpha(n)}{n}\operatorname{ess\,sup} \mathbb{E}[Z_\alpha U_\alpha|Y] = \sum_{a=1}^{K} q_\alpha \operatorname{ess\,sup} \mathbb{E}[Z_\alpha U_\alpha|Y] \\
&= \sum_{a=1}^{K} q_\alpha \mathbb{1}_{[p_\alpha(Y)>0]} \operatorname{ess\,sup} U_\alpha
\end{aligned}$$

For $x < \bar{x}(y)$, by the Weak Law of Large Numbers

$$\lim_{n\to\infty} \mathbb{P}(L_n \geq nx|y) = 1$$

For $x > \hat{x}(y)$ clearly

$$\lim_{n\to\infty} \mathbb{P}(L_n \geq nx|y) = 0$$

To calculate $\mathbb{P}(L_n \geq nx|y)$ for $\bar{x}(y) < x < \hat{x}(y)$ we notice that, by property 6, conditional on Y, the summands of L_n (i.e. $Z_{\alpha,i}U_{\alpha,i}$) are independent, but belong to K different distribution. Hence we are exactly in the setup of Corollaries 2 and 3. Moreover, by Lemma 1 and the mean value theorem, we know that for $\bar{x}(y) < x < \hat{x}(y)$ there exists a unique $\xi > 0$ with $\Lambda'(\xi) = x$.

2.2.1 Applying Cramér's Theorem

Let $\bar{x}(y) < x < \hat{x}(y)$. Then by Corollary 3

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(L_n \geq nx|y) = -\Lambda^*(x|y)$$

where $\Lambda^*(x|Y) := \sup\{\eta \in \mathbb{R} : \eta x - \Lambda(\eta|Y)\}$.

Observing that the last statement also holds for $x < \bar{x}(y)$ and $x > \hat{x}(y)$ we get by the law of total probability and the Laplace principle for almost all $x > 0$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_n \geq nx) &= \lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_y \mathbb{P}(Y = y)\mathbb{P}(L_n \geq nx|y)\Big) \\ &= \max_y(-\Lambda^*(x|y)) = \max_y \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_n \geq nx|y)\end{aligned}$$

The maximum on the right hand side will be attained for the value y of Y which models the "worst case scenario". Hence this estimation is unsatisfactory.

2.2.2 Applying Bahadur&Rao's Theorem

Let $\bar{x}(y) < x < \hat{x}(y)$. Then by Corollary 2

$$\lim_{n\to\infty} J_n(x|y)\mathbb{P}(L_n \geq nx|y) = 1$$

where $J_n(x|Y) = c_n\xi\sqrt{\Lambda''(\xi|Y)2\pi n}\exp(n\Lambda^*(x|Y))$ and $c_n := \exp\big(n(\Lambda_n(\xi|Y) - \Lambda(\xi|Y))\big)$. Now let

$$K_n(x|Y) := \begin{cases} 1 & \text{if } x < \bar{x}(Y) \\ \frac{1}{J_n(x|Y)} & \text{if } \bar{x}(Y) < x < \hat{x}(Y) \\ 0 & \text{if } x > \hat{x}(Y) \end{cases}$$

Then we have for almost all $0 < x < \max_y \hat{x}(y)$

$$\lim_{n\to\infty} \frac{\mathbb{P}(L_n \geq nx)}{K_n(x)} = 1$$

where $K_n(x) = \sum_y \mathbb{P}(Y = y)K_n(x|y)$.

Note that $K_n(x)$ does not only make sense in the limit statement, but is already a (very) good estimate of $\mathbb{P}(L_n \geq nx)$ for large n. Moreover it is given by an explicit formula making further calculation (for example sensitivity analysis) possible and easier. In particular the formula is flexible with respect to the distribution of Y, which has to be forecast.

References

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[2] Arthur Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, 2 ed., 1998.

[3] William Feller, *An introduction to probability theory and its applications*, vol. 2, 1966.