DISTANCES IN SCALE FREE NETWORKS AT CRITICALITY

STEFFEN DEREICH, CHRISTIAN MÖNCH, AND PETER MÖRTERS

Abstract: We look at preferential attachment networks in a critical window where the asymptotic proportion of vertices with degree at least k scales like $k^{-2}(\log k)^{2\alpha+o(1)}$ and show that two randomly chosen vertices in the same component of the graph have distance $\big(\frac{1}{1+\alpha}+o(1)\big)\frac{\log N}{\log\log N}$ in probability as the number N of vertices goes to infinity. By contrast the distance in a rank one model with the same asymptotic degree sequence is $\big(\frac{1}{1+2\alpha}+o(1)\big)\frac{\log N}{\log\log N}$. In the limit as $\alpha\to\infty$ we see the emergence of a factor two between the length of shortest paths as we approach the ultrasmall regime.

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1. Background and Motivation

Scale-free networks are characterised by the fact that, as the network size goes to infinity, the asymptotic proportion of nodes with degree at least k behaves like $k^{-\tau+1+o(1)}$ for some power law exponent τ. There are a number of mathematical models for scale-free networks, in the class of **rank-one models** the probability that two vertices are directly connected is asymptotically equivalent to the product of suitably defined weights w_v associated to the vertices v in a network $\mathcal{G}_N$ with vertex set $[N]:=\{1,\dots,N\}$. Examples of rank-one models are the *Chung-Lu model* where

$$\mathbb{P}(u\leftrightarrow v)=\frac{w_uw_v}{\sum_{i=1}^N w_i}\wedge 1,\qquad \text{for } u,v\in[N],$$

the *Norros-Reittu model* in which

$$\mathbb{P}(u\leftrightarrow v)=1-\mathrm{e}^{-\frac{w_uw_v}{\sum_{i=1}^N w_i}},\qquad \text{for } u,v\in[N], \tag{1.1}$$

where $(w_i)_{i=1}^N$ is a deterministic or random sequence of weights, and the *configuration model* in which each vertex is assigned a degree chosen randomly from a given degree distribution and the weights are the degrees themselves.

A popular alternative to rank one models are the **preferential attachment models** introduced by Barabási and Albert. The original *Barabási-Albert model* (see Bollobas et al. [BRST01] for a rigorous definition) is a dynamical network model in which new vertices connect to a fixed number of existing vertices with a probability proportional to their degree. In this model the power law exponent is always $\tau=3$. Recent variants introduced by van der Hofstad et al. [DHH10] and Dereich and Mörters [DM09], allow the connection probability to be proportional to a function of the degree and can therefore generate networks with

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variable power law exponent $\tau > 2$. Physicists have predicted that all these models of scale-free networks with the same power law exponent share essentially the same global topology, see for example [AB02].

In their seminal paper [BR04] Bollobás and Riordan show that two randomly chosen vertices in the original Barabasi-Albert model have a graph distance asymptotically equivalent to $\frac{\log N}{\log\log N}$. The same result holds for a variety of other models of scale-free networks when the asymptotic proportion of vertices with degree at least k scales like k^{-2}. Examples include the rank one models of Chung and Lu, of Norros and Reittu, inhomogeneous random graphs with a suitable choice of kernel, and the configuration model.

It was therefore believed that distances in preferential attachment models behave similarly to distances in the configuration model with similar degrees, see for example [HHZ07]. It thus came as a surprise when in [DMM12] we showed that in the ultrasmall regime, i.e. when the power law exponent is in the range $2 < \tau < 3$, distances in preferential attachment models are twice as long as in the rank one models above. This is due to the fact that two vertices of high degree in the preferential attachment model are much more likely to be connected by a path of length two, rather than a path of length one as in the rank one models.

It is the aim of the present paper to study the emergence of this factor two. Does the factor occur at a sharp threshold and if so where is this threshold? Or is there a smooth transition between the factors one and two in a suitably chosen critical window? To answer these questions we look at preferential attachment models in the framework of [DM09, DM13]. In these network models the attachment probabilities can be chosen as a nonlinear function of the vertex degree giving enough flexibility to generate varying asymptotic degree distributions. We study the case where the asymptotic proportion of nodes with degree at least k scales like $k^{-2}(\log k)^{2\alpha+o(1)}$, for some $\alpha > 0$, and compare with the Norros-Reittu model with i.i.d. weights whose degree sequence has the same tail behaviour. Our main result shows that typical distances in the preferential attachment networks are bigger by an asymptotic factor of $\frac{1+2\alpha}{1+\alpha}$, which converges to two as $\alpha \uparrow \infty$.

2. Statement of the main results

Our main result concerns the variant of the preferential attachment model introduced in [DM09], which has the advantage over other variants of remaining tractable even when the connection probability is a nonlinear function of the degree of the older vertex. To define the model precisely fix a concave function $f\colon \mathbb{N}_0 \to (0,\infty)$, which is called the *attachment rule*, and define a sequence of random graphs $(\mathcal{G}_N)_{N\in\mathbb{N}}$ in the following way:

(1) The initial graph $\mathcal{G}_1$ is a single vertex labelled 1.
(2) Given $\mathcal{G}_N$, the graph $\mathcal{G}_{N+1}$ is obtained by
  - adding a new vertex labelled $N+1$;
  - independently for any vertex with label $m \le N$ insert an edge between this vertex and the new vertex with probability
$$\frac{f(\mathcal{Z}[m,N])}{N} \wedge 1,$$
where $\mathcal{Z}[m,N] := \sum_{i=m+1}^{N} \mathbb{1}\{m \leftrightarrow i\}$ is the number of younger vertices connecting to i in $\mathcal{G}_N$.

If we orient all edges from the younger to the older vertex we can interpret $\mathcal{Z}[m,N]$ as the *indegree* of the vertex labelled i in the oriented graph derived from $\mathcal{G}_N$. Note however that throughout this paper we consider the graphs $\mathcal{G}_N$ as unoriented and the notions of connectivity and graph distance d_N taken in $\mathcal{G}_N$ are with reference to unoriented edges. For any potential edge $(v,w) \subset [N]^2$ with $v < w$ we write $v \leftrightarrow w$ if we wish to indicate that (v,w) is contained in $\mathcal{G}_N$. When it is convenient to stress the original orientation we write $v \leftarrow w$ or $w \rightarrow v$.

The following theorem identifies the class of attachment rules which produces typical distances of order $\frac{\log N}{\log\log N}$. It is the main result of this paper.

Theorem 1. *Let $(\mathcal{G}_N)_{N\in\mathbb{N}}$ be the sublinear preferential attachment model obtained from a concave attachment rule f satisfying*

$$f(k) = \frac{1}{2}k + \frac{\alpha}{2}\frac{k}{\log k} + o\Big(\frac{k}{\log k}\Big), \tag{2.1}$$

for some $\alpha > 0$. Consider two vertices U, V chosen independently and uniformly at random from the largest connected component $\mathcal{C}_N \subset \mathcal{G}_N$, then

$$d_N(U,V) = \Big(\frac{1}{1+\alpha} + o(1)\Big)\frac{\log N}{\log\log N} \quad \textit{with high probability as } N \to \infty.$$

It is shown in [DM09] that the asymptotic degree distribution in the preferential attachment graph $\mathcal{G}_N$ with the attachment rule given in (2.1) satisfies

$$\frac{1}{N}\sum_{v\in\mathcal{G}_N} \mathbb{1}\{\mathrm{degree}(v) \geq k\} = k^{-2}(\log k)^{2\alpha+o(1)} \quad \text{in probability.} \tag{2.2}$$

We contrast the result of Theorem 1 on the typical distances in the *preferential attachment model* with a result on the typical distances in the *Norros-Reittu model* with an i.i.d. weight sequence parametrised to obtain the same tail behaviour of the empirical degree distribution. We choose this model for definiteness but the result extends easily to other rank-one models, such as the Chung-Lu model, and to deterministic weight sequences with similar asymptotics.

To define the model, given a distribution on the positive reals we generate a sequence $W = (W_i)_{i=1}^\infty$ of i.i.d. random variables with this distribution. Let $L_N = \sum_{n=1}^N W_n$ denote the total weight of the vertices in $[N]$. For fixed N and given the weights $W_1, \dots, W_N$ we construct the random graph $\mathcal{H}_N = \mathcal{H}_N(W)$ with vertex set $[N]$ as follows:

- Between any two distinct vertices $v, w \in [N]$ the number of edges is Poisson distributed with parameter $\frac{W_v W_w}{L_N}$, independent of all other edges.
- Parallel edges are merged to obtain a simple graph.

Theorem 2. *Let $(\mathcal{H}_N)_{N\in\mathbb{N}}$ denote the Norros-Reittu model with weight distribution satisfying*

$$\mathbb{P}(W_1 \geq k) = k^{-2}(\log k)^{2\alpha+o(1)}, \tag{2.3}$$

for $\alpha > 0$. Consider two vertices U, V chosen independently and uniformly at random from the largest connected component $\mathcal{C}_N \subset \mathcal{H}$, then

$$d_N(U,V) = \Big(\frac{1}{1+2\alpha} + o(1)\Big)\frac{\log N}{\log\log N} \quad \textit{with high probability as } N \to \infty.$$

We observe that the characteristic difference in the typical distances between preferential attachment models and rank-one models in the ultrasmall phase does not occur suddenly at the phase transition, but arises gradually in a critical window. For networks with empirical degree distributions decaying as in (2.2) there is a factor of $\frac{1+2\alpha}{1+\alpha}$ between the typical distances in the two types of networks. This factor converges to two as we approach the ultrasmall regime by letting $\alpha \uparrow \infty$, and converges to one as we approach the linear case by letting $\alpha \downarrow 0$. A heuristic explanation for this transition is that in the preferential attachment model in the critical window the probabilities that two vertices of high indegree are connected directly or via a young connector vertex are on the same scale. Hence the asymptotical proportion of the transistions between vertices on a typical short path that use a connector, is a constant strictly between zero and one. This constant turns out to be $\frac{\alpha}{1+\alpha}$ and this yields a factor $1+\frac{\alpha}{1+\alpha}$ by which the length of shortest paths in the preferential attachment model exceed that in the rank-one models.

Qualitatively different behaviour for the preferential attachment and rank-one model class can also be observed when studying robustness of the giant component under targeted attack, see Eckhoff and Mörters [EM13], or in the behaviour of the size of the giant component near criticality, see forthcoming work of Eckhoff, Mörters and Ortgiese [EMO15].

3. Proof of lower bounds – preferential attachment

Lower bounds for average distances are proved using a first moment method. To this end, Section 3.1 provides bounds for expected degrees in the preferential attachment model, which are used in Section 3.2 to prove the lower bound in Theorem 1. In all subsequent proofs a subscript number on a constant refers to the place where it is defined, e.g. $C_{1.23}$ is the constant introduced in Lemma 1.23.

3.1. **Degree asymptotics for preferential attachment.** It follows directly from the definition of the preferential attachment graphs, that the evolutions $(\mathcal{Z}[1,n])_{n\geq 1}, (\mathcal{Z}[2,n])_{n\geq 2},\dots$ form an independent collection of Markov chains which describe the network completely. We derive lower and upper bounds for $\mathbb{E} f(\mathcal{Z}[m,n])$. For conciseness in the formulation of later results, we allow $(\mathcal{Z}[m,n])_{n\geq m}$ to start in any integer $k\in\mathbb{N}$ and denote the resulting distribution by $\mathbb{P}^k$, and its expectation by $\mathbb{E}^k$.

Lemma 3.1. *Let $k,m\in\mathbb{N}$, $\mathcal{Z}[m,m]=k$ be fixed and define, for $n\geq m$,*

$$X(n)=\frac{f(\mathcal{Z}[m,n])}{\xi(m,n)} \quad \text{and} \quad Y(n)=\frac{f(\mathcal{Z}[m,n])^2+\frac12 f(\mathcal{Z}[m,n])}{\frac{n}{m}},$$

where $\xi(m,n)$ is given by

$$\xi(m,n)=\prod_{i=m}^{n-1}\Big(1+\frac{1}{2i}\Big)=\frac{\Gamma(n+\frac12)\Gamma(m)}{\Gamma(m+\frac12)\Gamma(n)}.$$

Then $X=(X(n))_{n\geq m}$ and $Y=(Y(n))_{n\geq m}$ are submartingales. If f is affine, then they are martingales.

Proof. Fix $n\geq m$ and let $\Delta\mathcal{Z}[m,n]=\mathcal{Z}[m,n+1]-\mathcal{Z}[m,n]$. The martingale property of X for an affine attachment rule $f(x)=\frac12 x+\beta$ follows immediately from

$$\begin{aligned}\mathbb{E}^k\big[f(\mathcal{Z}[m,n+1])\big|\mathcal{Z}[m,n]\big]&=\mathbb{E}^k\big[f(\mathcal{Z}[m,n])+\tfrac12\mathbb{1}\{n+1\to m\}\big|\mathcal{Z}[m,n]\big]\\&=\mathbb{E}^k\big[f(\mathcal{Z}[m,n])\big|\mathcal{Z}[m,n]\big]+\tfrac12\mathbb{E}\big[\tfrac{f(\mathcal{Z}[m,n])}{n}\big|\mathcal{Z}[m,n]\big]\\&=\big(1+\tfrac{1}{2n}\big)f(\mathcal{Z}[m,n]).\end{aligned}\tag{3.1}$$

The corresponding calculation for Y is performed in complete analogy to (3.1), we obtain

$$\mathbb{E}\big[f(\mathcal{Z}[m,n+1])^2\big|\mathcal{Z}[m,n]\big]=\big(1+\tfrac1n\big)f(\mathcal{Z}[m,n])^2+\tfrac{1}{4n}f(\mathcal{Z}[m,n]),$$

and thus

$$\begin{aligned}\tfrac{n+1}{m}\mathbb{E}\big[Y(n+1)\big|\mathcal{Z}[m,n]\big] &= \big(1+\tfrac{1}{n}\big)f(\mathcal{Z}[m,n])^2 + \tfrac{1}{4n}f(\mathcal{Z}[m,n]) + \tfrac{1}{2}\big(1+\tfrac{1}{2n}\big)f(\mathcal{Z}[m,n])\\ &= \big(1+\tfrac{1}{n}\big)\tfrac{n}{m}Y(n).\end{aligned}$$

Division by $(1+\frac{1}{n})\frac{n}{m} = \frac{n+1}{m}$ now yields the martingale property. For strictly concave f, we have $\Delta f(i) = f(i+1) - f(i) > \frac{1}{2}$, for all $i \in \mathbb{N}$, and the equalities in the above calculations turn into inequalities which yields the submartingale property. □

By Lemma 3.1, for all $n \geq m \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\xi(m,n) = \prod_{i=m}^{n-1}\Big(1+\frac{1}{2i}\Big) \in \Big[\sqrt{\frac{n}{m}}, (1+\delta(m))\sqrt{\frac{n}{m}}\Big], \tag{3.2}$$

where $\delta(m)$ can be chosen such that $\lim_{m\to\infty}\delta(m) = 0$. In the affine case $\xi(m,n) = \frac{1}{f(k)}\,\mathbb{E}^k f(\mathcal{Z}[m,n])$, in particular the *score* $\xi(m,N)$ of a vertex m is asymptotically proportional to its expected degree at time N. For the deviation from the affine case we introduce the notation

$$\psi^k(m,n) := \frac{\mathbb{E}^k f(\mathcal{Z}[m,n])}{\xi(m,n)}. \tag{3.3}$$

If the graph size N is fixed, we write $\xi(m), \psi^k(m)$ for $\xi(m,N), \psi^k(m,N)$. If $V \subset [N]$ is a set of vertices we also use the shorthand notation $\xi^n(V) := \sum_{v\in V}\xi(v)^n$ for the score and its higher moments. Determining the magnitude of ψ^k is the first step towards the proof of Theorem 1. As we will see later, it suffices to study the special case

$$f(k) = \frac{k}{2} + \frac{\alpha}{2}\frac{k}{1\vee\log k} + \beta, \qquad \text{for } k \geq 0, \tag{3.4}$$

with $\alpha \geq 0$ and $\beta = f(0) > 0$.

Proposition 3.2 (First and second moment upper bound)**.** *Let f be an attachment rule of the form* (3.4)*. Then, for any $k \in \mathbb{N}$, there exist constants $C = C(\alpha,k), C' = C'(\alpha,k)$ such that for all pairs $m,n \in \mathbb{N}$ with $n \geq m$,*

$$\mathbb{E}^k f(\mathcal{Z}[m,n]) \leq C\sqrt{\frac{n}{m}}\big(1\vee\log\tfrac{n}{m}\big)^\alpha$$

and

$$\mathbb{E}^k f(\mathcal{Z}[m,n])^2 \leq C'\frac{n}{m}\big(1\vee\log\tfrac{n}{m}\big)^{2\alpha}.$$

Proposition 3.3 (First moment lower bound)**.** *Let f be as in* (3.4)*. Then, for any $k \in \mathbb{N}$ there exists a constant $c = c(\alpha,k) > 0$, such that for all pairs $m,n \in \mathbb{N}$ with $n \geq m$,*

$$\mathbb{E}^k f(\mathcal{Z}[m,n]) \geq c\sqrt{\frac{n}{m}}\big(1\vee\log\tfrac{n}{m}\big)^\alpha.$$

We note that the two propositions together imply that there are constants $0 < c' \leq C'$ depending only on α and k, such that

$$c'\,\big(1\vee\log\tfrac{n}{m}\big)^\alpha \leq \psi^k(m,n) \leq C''\,\big(1\vee\log\tfrac{n}{m}\big)^\alpha \tag{3.5}$$

To prove Proposition 3.2 and Proposition 3.3 we need two auxiliary statements, which are straightforward consequences of our assumptions on f.

Lemma 3.4. *Let f be a concave attachment rule and g be given by*

$$g(x)=\frac{x}{\log(f^{-1}(x))},\qquad \text{for } x\in\{f(k),k\in\mathbb{N}\},$$

then there is $K=K(f)\in\mathbb{N}$, such that g is concave on $\{f(k),k\geq K\}$.

Proof. By interpolation we can assume that f is twice differentiable on $(0,\infty)$ with existing right derivative in 0. Let e denote the inverse of f, which is a well defined convex function, since f is increasing and concave. The second derivative of g is given by

$$g''(x)=\frac{x(e'(x))^2(\log e(x)+2)-e(x)\log e(x)(xe''(x)+2e'(x))}{e(x)^2(\log e(x))^3}, \tag{3.6}$$

for $x\in[0,\infty)$. To see that $g''(x)\leq 0$ for large x, we note that $e''(x)\geq 0$ and $e'(0)\leq e'(x)(\lim_{k\to\infty}\Delta f(k))^{-1}$. As $e(x)$ is bounded below by $x-1$, the numerator in (3.6) is non-positive for sufficiently large x. $\square$

Lemma 3.5. *Let f satisfy condition (3.4) with $\alpha>0$ and set $\varphi(x)=\sum_{i=0}^{x-1}\frac{1}{f(i)}$. Then,*

(i) the linear interpolation $\varphi^{-1}\colon[\frac{1}{f(0)},\infty)\longrightarrow[0,\infty)$ of the inverse of φ exists and is strictly monotone, in particular, for $x\geq\frac{1}{f(0)}$ and $k\in\mathbb{N}$,

$$\varphi^{-1}(x)\geq k,\quad \text{if } x\geq\varphi(k);$$

(ii) there are constants $c,C>0$, only depending on f, such that, for all $k\in\mathbb{N}$,

$$\frac{1}{f(0)}\vee\big(2\log_+k-2\alpha\log\log_+k-c\big)\leq\varphi(k)\leq 2\log_+(k-1)-2\alpha\log\log_+(k-1)+C,$$

where $\log_+(k)=\log(k\vee 1)$ and $\log\log_+(k)=\log\log(k\vee \mathrm{e})$.

Proof. For (i) note, that the attachment rule f is positive and strictly increasing, which implies that $\Delta\varphi=1/f>0$ is strictly decreasing. Thus φ is concave and strictly increasing, hence its inverse is well defined, convex, strictly increasing and $\varphi^{-1}(x)=k$, if $x=\sum_{i=0}^{k-1}1/f(i)$, and the claimed monotonicity is inherited by the linear interpolation. To show (ii), we note that $\varphi(k)\geq\frac{1}{f(0)}$ and that $1/f(i)=2/i-2\alpha/i\log i+1/i(\log i)^2$, from which the statement follows by summation. $\square$

Proof of Proposition 3.2. We start with the first moment and decompose

$$f(\mathcal{Z}[m,n])=f(k)+\sum_{s=m}^{n-1}\frac{f(\mathcal{Z}[m,s])}{s}\Delta f(\mathcal{Z}[m,s])+M_m^f(n), \tag{3.7}$$

where M_m^f is a martingale which can be easily seen by a direct calculation or application of general results about martingales derived from Markov chains, e.g. [BL12, Lemma 2.1]. Taking expectations we obtain the recursion

$$\mathbb{E}^kf(\mathcal{Z}[m,n+1])=\mathbb{E}^kf(\mathcal{Z}[m,n])+\mathbb{E}^k\frac{f(\mathcal{Z}[m,n])\Delta f(\mathcal{Z}[m,n])}{n}. \tag{3.8}$$

Fix k_0 such that $f(k)>\mathrm{e}^2$ and $\Delta f(k)\leq\frac{1}{2}+\frac{\alpha}{2\log f(k)}$ for all $k\geq k_0$. Then, for $k\geq k_0$,

$$\mathbb{E}^kf(\mathcal{Z}[m,n])\Delta f(\mathcal{Z}[m,n])\leq\frac{1}{2}\mathbb{E}^kf(\mathcal{Z}[m,n])+\frac{\alpha}{2}\mathbb{E}^k\frac{f(\mathcal{Z}[m,n])}{\log f(\mathcal{Z}[m,n])}.$$

The function $x\mapsto\frac{x}{(\log x)}$ is concave on (e^2,∞), and we apply Jensen's inequality to the second term in this sum and obtain

$$\mathbb{E}^kf(\mathcal{Z}[m,n])\Delta f(\mathcal{Z}[m,n])\leq\frac{1}{2}\mathbb{E}^kf(\mathcal{Z}[m,n])+\frac{\alpha}{2}\frac{\mathbb{E}^kf(\mathcal{Z}[m,n])}{\log\mathbb{E}^kf(\mathcal{Z}[m,n])}.$$

Applying this bound to the right hand side of (3.8) yields, after division by $\mathbb{E}^k f(\mathcal{Z}[m,n])$,

$$\frac{\mathbb{E}^k f(\mathcal{Z}[m,n+1])}{\mathbb{E}^k f(\mathcal{Z}[m,n])} \le 1 + \frac{1}{2n} + \frac{\alpha}{2n \log \mathbb{E}^k f(\mathcal{Z}[m,n])}. \tag{3.9}$$

We can apply the lower bound in (3.2) to bound the denominator of the last term from below by $n\big(1 \vee \log \frac{n}{m}\big)$ to get

$$\frac{\mathbb{E}^k f(\mathcal{Z}[m,n+1])}{\mathbb{E}^k f(\mathcal{Z}[m,n])} \le 1 + \frac{1}{2n} + \frac{\alpha}{n(1 \vee \log \frac{n}{m})}. \tag{3.10}$$

Iterating both sides of (3.10) in n then yields

$$\mathbb{E}^k f(\mathcal{Z}[m,n]) \le f(k) \prod_{i=m}^{n-1} \Big(1 + \frac{1}{2i} + \frac{\alpha}{i(1 \vee \log \frac{i}{m})}\Big),$$

and using the inequality $1 + x \le \mathrm{e}^x$ we get

$$\mathbb{E}^k f(\mathcal{Z}[m,n]) \le f(k) \exp\Big(\sum_{i=m}^{n-1} \frac{1}{2i} + \sum_{i=m}^{n-1} \frac{\alpha}{i(1 \vee \log \frac{i}{m})}\Big),$$

which implies

$$\mathbb{E}^k f(\mathcal{Z}[m,n]) \le f(k) \exp\Big[\frac{1}{2}\sum_{i=m}^{n-1} \frac{1}{i} + \alpha\Big(\sum_{i=m}^{\lceil \mathrm{e}m\rceil -1} \frac{1}{ik} + \sum_{i=\lceil \mathrm{e}m\rceil}^{n-1} \frac{1}{i \log \frac{i}{m}}\Big)\Big]. \tag{3.11}$$

The exponential of $\frac{1}{2}\sum_{i=m}^{n-1} \frac{1}{i}$ is less than $D\sqrt{\frac{n}{m}}$, for some constant D. To handle the second expression in the exponential we observe that $\sum_{i=m}^{\lceil \mathrm{e}m\rceil -1} \frac{1}{ik} \le \frac{11}{6k}$ and

$$\sum_{i=\lceil \mathrm{e}m\rceil}^{n-1} \frac{1}{i \log \frac{i}{m}} \le \int_{\mathrm{e}m}^{n} \frac{\frac{1}{m}}{\frac{s}{m}\log \frac{s}{m}}\, \mathrm{d}s + C'' = \int_{\mathrm{e}}^{\frac{n}{m}} \frac{1}{x \log x}\, \mathrm{d}x + C'' = \log\log \tfrac{n}{m} + C'',$$

for some absolute constant C''. Applying these estimates to (3.11) we arrive at

$$\mathbb{E}^k \mathcal{Z}[m,n] \le f(k)\mathrm{e}^{\frac{11\alpha}{6k}+C''} D\sqrt{\frac{n}{m}}\big(1 \vee \log \tfrac{n}{m}\big)^{\alpha},$$

proving the desired bound for $C(\alpha,k) = f(k)\mathrm{e}^{\frac{11\alpha}{6k}+C''}D$.

It remains to deduce the bound for the second moment. We argue as for the first moment, but apply the decomposition (3.7) to the function f^2, i.e. we obtain

$$\mathbb{E}^k f(\mathcal{Z}[m,n])^2 = f(k)^2 + \sum_{s=m}^{n-1} \mathbb{E}^k \frac{f(\mathcal{Z}[m,s])\Delta f(\mathcal{Z}[m,s])^2}{s}.$$

Since f is nondecreasing, we find that $\Delta f(k)^2 \le f(k+1)2\Delta f(k)$ and thus

$$\mathbb{E}^k f(\mathcal{Z}[m,n])^2 \le f(k)^2 + \sum_{s=m}^{n-1} \mathbb{E}^k \frac{2f(\mathcal{Z}[m,s]+1)^2\Delta f(\mathcal{Z}[m,s])}{s} =: E(m,n).$$

The function $E(m,n)$ can be bounded in the same fashion as the first moment, we obtain

$$\frac{E(m,n+1)}{E(m,n)} \le 1 + \frac{1}{n} + \frac{2\alpha}{n(1 \vee \log \frac{n}{m})},$$

which implies $E(m,n) \le C'(\alpha,k)\frac{n}{m}\big(\log \frac{n}{m}\big)^{2\alpha}$, and hence the second moment bound. □

Proof of Proposition 3.3. We only need to focus on the lower bound for $k = 0$ and begin with the observation that the concavity condition on f implies that

$$\mathbb{E}f(\mathcal{Z}[m,n]) \geq f(0) + \tfrac{1}{2}\mathbb{E}\mathcal{Z}[m,n]. \tag{3.12}$$

We can represent $\varphi(\mathcal{Z}[m,n]) = \sum_{i=m}^{n-1} \frac{1}{i} + M_n$, where $(M_n)_{n\geq m}$ is a martingale, see [DM09, Lemma 2.1], hence

$$\mathbb{E}\varphi(\mathcal{Z}[m,n]) = \frac{1}{f(0)} + \sum_{i=m}^{n-1} \frac{1}{i}.$$

Using concavity of φ, Jensen's inequality implies that

$$\varphi(\mathbb{E}\mathcal{Z}[m,n]) \geq \frac{1}{f(0)} + \sum_{i=m}^{n-1} \frac{1}{i},$$

which yields, together with the upper bound on φ from Lemma 3.5,

$$(C + 2\log \mathbb{E}\mathcal{Z}[m,n] - 2\alpha \log\log \mathbb{E}\mathcal{Z}[m,n]) \vee 0 \geq \log \tfrac{n}{m}$$

for some suitably chosen constant $C > 0$. This yields $\mathbb{E}\mathcal{Z}[m,n] \geq d\sqrt{\frac{n}{m}}(\log \mathbb{E}\mathcal{Z}[m,n] \vee 0)^\alpha$ for some small constant $d > 0$ and combining the last inequality with (3.12) we obtain

$$\mathbb{E}f(\mathcal{Z}[m,n]) \geq f(0) + \frac{d}{2}\sqrt{\frac{n}{m}}(\log \mathbb{E}\mathcal{Z}[m,n] \vee 0)^\alpha.$$

The expectation on the right can be bounded below by the expectation in the affine case, for which a lower bound is implicit in (3.2). For all sufficiently large $n > m$ we get

$$\mathbb{E}f(\mathcal{Z}[m,n]) \geq f(0) + c'\sqrt{\frac{n}{m}}(1 \vee \log \tfrac{n}{m})^\alpha$$

for some $c' > 0$ and a further adjustment of the constant, which only depends on the value $f(0)$, yields the statement of the proposition. □

We close this section with two very intuitive stochastic domination results from [DM13] which are instrumental in the proof of Theorem 1.

Lemma 3.6 (Stochastic domination I, [DM13, Lemma 2.9])**.** *Let f be concave and fix integers $m < n_1 < \cdots < n_i$. The process $(\mathcal{Z}[m,n])_{n\geq m}$ conditioned on the event $\{\Delta\mathcal{Z}[m,n_j] = 0, j = 1,\ldots,i\}$ is stochastically dominated by the unconditioned process.*

Proof. See [DM13, p. 18]. □

Lemma 3.7 (Stochastic domination II, [DM13, Lemma 2.10])**.** *Let f be concave and fix $i,k \in \mathbb{N}$. For integers $n_i > \cdots > n_1 > m > k+i$ there is a coupling of the process $(\mathcal{Z}[m,l])_{l\geq m}$ started in $\mathcal{Z}[m,m] = k$ and conditioned on $\{\Delta\mathcal{Z}[m,n_j] = 1\ \forall j \in \{1,\ldots,i\}\}$ and the unconditional process $(\mathcal{Z}[m,l])_{l\geq m}$ started in $\mathcal{Z}[m,m] = k+i$ such that for the coupled versions $(\bar{\mathcal{Z}}^{(c)}[m,l], \bar{\mathcal{Z}}^{(u)}[m,l])_{l\geq m}$ one has*

$$\Delta\bar{\mathcal{Z}}^{(c)}[m,l] \leq \Delta\bar{\mathcal{Z}}^{(u)}[m,l] + \sum_{j=1}^{i} \mathbb{1}\{l = n_j\}.$$

Thus the unconditioned process initiated in $k+i$ dominates the process initiated at k and conditioned to have jumps at time $n_1,\ldots,n_i$.

Proof. The case $i = 1$ is the original statement [DM13, Lemma 2.10] and proven there. The general case follows by a straightforward induction argument. □

3.2. **Lower bounds for distances.** The first moment estimates of the previous section now yield lower bounds on the typical distances in a straightforward manner under the assumption of bounded correlation for edges along any self-avoiding path.

Lemma 3.8 (First order lower bound on distances)**.** *Let $\mathcal{G}_N$ be a random graph with vertex set $[N]$ and assume that there are $\kappa_N \geq 0$ and $\Psi_N \geq 0$, such that, for any self-avoiding path $P = (v_0, \dots, v_l)$, we have*

$$\mathbb{P}(P \subset \mathcal{G}_N) \leq \kappa_N^l \prod_{j=0}^{l-1} \mathbb{P}(v_j \leftrightarrow v_{j+1}) \tag{3.13}$$

and

$$\mathbb{P}(v \leftrightarrow w) \leq \frac{\Psi_N}{\sqrt{vw}}, \quad \textit{for all } v, w \in [N], \tag{3.14}$$

where

$$\lim_{N\to\infty} \frac{\log \Psi_N + \log \kappa_N}{\log N} = 0. \tag{3.15}$$

Then, for uniformly chosen vertices $U, V \in \mathcal{G}_N$,

$$\lim_{N\to\infty} \mathbb{P}\Big(d_N(U,V) \geq \Big\lceil \frac{\log N}{\log\log N + \log \Psi_N + \log \kappa_N} \Big\rceil\Big) = 1.$$

Proof. We first observe that for any positive sequence $(a_i)_{i=0}^\infty$ satisfying $\frac{a_{i+1}}{a_i} \geq 1 + \delta$, for all $i \geq 0$ and some fixed $\delta > 0$, we can find a constant $C > 0$ with

$$\sum_{i=0}^K a_i \leq C a_K, \quad \text{for all } K \in \mathbb{N}. \tag{3.16}$$

Let $1 \leq l \leq L = L(N) = \lfloor \frac{\log N}{\log\log N + \log(\kappa_N \Psi_N)} \rfloor$ and $P = (v_0, \dots, v_l)$ be self-avoiding. Assumptions (3.13) and (3.14) imply that

$$\mathbb{P}(P \subset \mathcal{G}_N) \leq \kappa_N^l \prod_{j=0}^{l-1} \frac{\Psi_N}{\sqrt{v_j v_{j+1}}} \leq \frac{\big(\kappa_N \Psi_N\big)^l}{\sqrt{v_0 v_l}} \prod_{j=1}^{l-1} \frac{1}{v_j}.$$

For $v, w \in [N]$ and $\mathcal{P}_l(v,w)$ denoting the set of self-avoiding paths of length l from v to w,

$$\begin{aligned}
\mathbb{P}(d_N(v,w) \leq L) &\leq \sum_{l=1}^{L} \sum_{(v_0,\dots,v_l) \in \mathcal{P}_l(v,w)} \frac{\big(\kappa_N \Psi_N\big)^l}{\sqrt{vw}} \prod_{j=1}^{l-1} \frac{1}{v_j} \\
&\leq \sum_{l=1}^{L} \frac{\big(\kappa_N \Psi_N\big)^l}{\sqrt{vw}} \Big(\sum_{j=1}^{N} \frac{1}{j}\Big)^{l-1} \leq \frac{1}{\sqrt{vw}} \sum_{l=1}^{L} \kappa_N^l \Psi_N^l (\log N)^{l-1}.
\end{aligned}$$

The terms in the last sum grow at least exponentially in l for all sufficiently large N, which implies by (3.16) the existence of an independent constant $C > 0$ such that

$$\mathbb{P}(d_N(v,w) \leq L) \leq C \frac{\big(\kappa_N \Psi_N \log N\big)^L}{\sqrt{vw} \log N}. \tag{3.17}$$

For any $\varepsilon \in (0,1)$, the probability that one of the vertices U, V is smaller than $\frac{\varepsilon}{3}N$ is bounded by $\frac{2}{3}\varepsilon$ and thus using (3.17) on the complement of this event results in

$$\begin{aligned}\mathbb{P}(d_N(V,W) \le L) &\le \sum_{v,w \ge \frac{\varepsilon}{3}N} \mathbb{P}(d_N(v,w) \le L)\mathbb{P}(V = v, W = w) + \frac{2\varepsilon}{3} \\ &\le 3C\frac{\big(\kappa_N \Psi_N \log N\big)^L}{\varepsilon N \log N} + \frac{2\varepsilon}{3} = \frac{3C}{\varepsilon \log N}\mathrm{e}^{L[\log\log N + \log(\kappa_N\Psi_N)] - \log N} + \frac{2\varepsilon}{3} \\ &\le \frac{3C}{\varepsilon \log N} + \frac{2\varepsilon}{3},\end{aligned}$$

and the proof is complete. □

The lower bounds on the distances in Theorem 1 can now be obtained by verifying the assumptions of Lemma 3.8.

Proposition 3.9 (Lower bounds for PA). *The preferential attachment model $\mathcal{G}_N$ with attachment rule f of the form* (2.1) *satisfies*

$$\lim_{N\to\infty} \mathbb{P}\Big(d_N(U,V) \ge \Big(\frac{1}{1+\alpha} - \delta\Big)\frac{\log N}{\log\log N}\Big) = 1,$$

for every $\delta > 0$ and independently and uniformly chosen vertices $U, V \in \mathcal{G}_N$.

Proof. Let $P = (v_0, \dots, v_n)$ be a self-avoiding path along vertices in $[N]$. By definition of the preferential attachment mechanism $\mathbb{P}(P \subset \mathcal{G}_N)$ can be decomposed into factors of the form $\mathbb{P}(u \to v \leftarrow w)$ and $\mathbb{P}(u \to v)$. If $v < u < w$, then $(\mathcal{Z}[v,n])_{n \ge v}$ conditional on $\mathcal{Z}[v,v] = 1$ stochastically dominates $(\mathcal{Z}[v,n]_{n\ge v}$ conditional on $\mathcal{Z}[v,u] = 1$ by Lemma 3.7 and hence

$$\mathbb{P}(\Delta\mathcal{Z}[v,w] = 1 | \Delta\mathcal{Z}[v,u] = 1) \le \mathbb{P}^1(\Delta\mathcal{Z}[v,w] = 1).$$

We obtain

$$\mathbb{P}(u \to v \leftarrow w) = \mathbb{P}(\Delta\mathcal{Z}[v,w] = \Delta\mathcal{Z}[v,u] = 1) \le \mathbb{P}(\Delta\mathcal{Z}[v,u] = 1)\mathbb{P}^1(\Delta\mathcal{Z}[v,w] = 1),$$

and in combination with Proposition 3.2 this shows that the edge correlation bound (3.13) is satisfied with $\kappa_N = \frac{C_{3.2}(\alpha,1)}{C_{3.2}(\alpha,0)}$. The bound (3.14) holds for $\Psi_N = C_{3.2}(\alpha,0)(\log N)^\alpha$, in the case where the attachment rule f is of the form (3.4). For such f the distance bound follows therefore for any choice of $\delta \in (0, \frac{1}{1+\alpha})$ immediately from Lemma 3.8.

For f of the more general form (2.1), we note that $\bar{f} \ge f$ implies that the respective networks satisfy $\bar{\mathcal{G}}_N \ge \mathcal{G}_N$ stochastically for all $N \in \mathbb{N}$, where $\ge$ is the partial order given by inclusion on the edge sets of graphs with the same vertex set, so that distances in $\mathcal{G}_N$ dominate those in $\bar{\mathcal{G}}_N$. By (2.1), for every $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}_0$,

$$f(k) \le f(k_0) + \frac{k}{2} + \frac{\alpha+\varepsilon}{2}\frac{k}{1 \vee \log k} =: \bar{f}(k).$$

Choosing ε suitably in dependence of δ thus allows to deduce the bound for general f from the special case treated in the previous paragraph. □

4. Proof of upper bounds – preferential attachment

For the upper bound we need to construct a path connecting two vertices by a short path. Starting from each of these vertices such a path has three phases, an *initial phase* in which we determine whether the vertex is in the giant component and, if it is, connect it to a vertex of large indegree, a *main phase* in which we connect the vertex of large indegree to a vertex in an inner core of highly connected hubs, and a *final phase* in which the two hubs thus obtained are connected to each other. The length of the path is asymptotically equal to sum of the number of steps in the two main phases. Sections 4.1 to 4.3 are devoted to the study of the three phases, and the proof of Theorem 1 is completed in Section 4.4.

4.1. **Local approximation results – initial phase.** A *configuration* $\mathbf{e}$ associates with every vertex a state in the set {veiled, active, dead}, and with every potential edge a state in the set $\{0, 1, \text{unknown}\}$, the state unknown capturing the absence of the information whether an edge is contained in $\mathcal{G}_N$ or not. The graph associated with a configuration consists of the vertex set $[N]$ and all edges in state 1. The score of a configuration is the cummulative score of all active vertices in the configuraton.

We now describe the *exploration process* that we follow in the initial phase as well as the main phase. Its definition uses a non-increasing sequence $(\ell_k)_{k\in\mathbb{N}}$ of truncation levels, which are set to $\ell_k = 1$, for all $k \in \mathbb{N}$, in the initial phase. The exploration is an inhomogeneous Markov chain $(\mathcal{E}_k)_{k\in\mathbb{N}}$ on the space of configurations, which we define on the probability space associated with the random graph $\mathcal{G}_N$. We assume that we start with an initial configuration $\mathcal{E}_0$, and the graph associated with this configuration is a tree.

In the kth exploration step we go through all active vertices in $\mathcal{E}_{k-1}$, starting with the vertex of smallest label and proceeding in increasing order of labels until all active vertices are treated. For each such vertex v we

(1) inspect all potential edges connecting v to veiled vertices in $\{\ell_k, \dots, N\}$;
(2) If the edge does not exist in $\mathcal{G}_N$ its state becomes 0 and the veiled vertex remains so;
(3) If it does exist in $\mathcal{G}_N$ its state becomes 1 and the veiled vertex is declared *pre-active.*

Once all active vertices are explored, they are declared dead, the pre-active vertices are declared active and the exploration step ends. Note that, if we start with a configuration associated with a tree, the configuration at the end of an exploration step will again be associated with a tree. We call such configurations *proper.* The sets of active, veiled and dead vertices of $\mathbf{e}$ are denoted by $\text{active}(\mathbf{e}), \text{veiled}(\mathbf{e})$ and $\text{dead}(\mathbf{e})$, respectively.

The following proposition (and nothing else in this paper) relies on a coupling of local neighbourhoods in $\mathcal{G}_N$ with the 'idealised neighbourhood tree' introduced in [DM13, Section 1.3]. The probability that this tree is infinite is denoted by $p(f)$. It coincides with the asymptotic proportion of vertices in the connected component of a uniformlly chosen vertex, and hence with the probability that such a vertex is in the giant component.

Proposition 4.1. *Suppose $U \in \mathcal{G}_N$ is uniformly chosen, determining an initial configuration in which U is active, all other vertices are veiled and all edges are in state unknown. Denote by $\xi(V) := \sum_{v\in V} \xi(v)$ the score associated with a set $V \subset [N]$ of vertices. Given $\varepsilon > 0$ and $s_0 > 0$ there exists $k_0 = k_0(s_0, \varepsilon) \in \mathbb{N}$, such that, for sufficiently large N, we have*

$$\mathbb{P}\big(\exists k \le k_0, A \subset \text{active}(\mathcal{E}_k) :\ \xi(A) \ge s_0\, \xi(\min A)\big) \ge p(f) - \varepsilon.$$

As the proof of Proposition 4.1 is obtained by application of the results of [DM13] and is therefore not self-contained we defer it to Appendix A.

4.2. **Score growth – main phase.** Our next goal is to fix a sequence $(\ell_k)_{k\ge 1}$ which guarantees that the score of encountered configurations during an exploration of the giant component grows with high probabilty at a certain deterministic rate. We rely on a careful analysis of the exploration process and the following concentration inequality.

Lemma 4.2 (Lower tail bound for independent sums, [CL06, Theorem 2.7])**.** *Let I be a finite set and $(X_i)_{\in I}$ be independent, nonnegative random variables. Then, for any $\lambda > 0$,*

$$\mathbb{P}\Big(\sum_{i\in I} X_i \le \sum_{i\in I} \mathbb{E}X_i - \lambda\Big) \le \mathrm{e}^{-\frac{\lambda^2}{2\sum_{i\in I}\mathbb{E}X_i^2}}.$$

We start the main phase in a proper configuration $\mathcal{E}_0$ with the property that the score of the set A of active vertices in the configuration satisfies $\xi(A) \ge s_0\xi(\min A)$ from some s_0 to be specified later. From this initial configuration we restart the exploration process $(\mathcal{E}_k : k \in \mathbb{N})$ using a new truncation sequence $(\ell_k)_{k\in\mathbb{N}}$. As before, each $\mathcal{E}_k$ is a proper configuration. As we gradually obtain more information about $\mathcal{G}_N$, we control the correlation between discovered edges using the following two lemmas.

Lemma 4.3 (Lower bound for conditional jump probabilities)**.** *For every $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ and a constant $C(m) > 0$ such that for every $n_0 \le n \le N$, and disjoint sets $I_0, I_1 \subset \{n, \dots, N-1\}$ with $\#I_1 \le m-1$ and*

$$\xi(n)\xi(I_0) \le C(m)\,\frac{N}{2\psi^m(n_0,N)}, \tag{4.1}$$

the events $A_i := \{\Delta\mathcal{Z}[n,j] = \mathbb{1}\{i=1\}$ for all $j \in I_i\}$, for $i \in \{0,1\}$, satisfy

$$\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_0,A_1) \ge \tfrac{1}{2}\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1) \quad \textit{for all } j \in \{n,\dots,N-1\}\setminus I_0.$$

Proof. Let $j \in \{n,\dots,N-1\}\setminus I_0$. We have

$$\begin{aligned}\mathbb{P}(\Delta\mathcal{Z}[n,j] = 1,&\Delta\mathcal{Z}[n,k]=0\ \forall k\in I_0|A_1)\\ &= \mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1) - \mathbb{P}(\Delta\mathcal{Z}[n,j]=1,\ \exists k\in I_0 : \Delta\mathcal{Z}[n,k]=1|A_1)\\ &\ge \mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1) - \sum_{k\in I_0}\mathbb{P}(\Delta\mathcal{Z}[n,j]=\Delta\mathcal{Z}[n,k]=1|A_1).\end{aligned} \tag{4.2}$$

The last sum can be rewritten

$$\begin{aligned}\sum_{k\in I_0}\mathbb{P}(\Delta\mathcal{Z}[n,j] =&\Delta\mathcal{Z}[n,k]=1|A_1)\\ &= \mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1)\sum_{k\in I_0}\mathbb{P}(\Delta\mathcal{Z}[n,k]=1|A_1,\Delta\mathcal{Z}[n,j]=1).\end{aligned} \tag{4.3}$$

Applying Lemma 3.7, (3.5) and (3.2) yield a constant $C(m)$ such that, for all $k \in I_0$,

$$\begin{aligned}\mathbb{P}(\Delta\mathcal{Z}[n,k]=1|A_1,\Delta\mathcal{Z}[n,j]=1) \le\ & \mathbb{P}^m(\Delta\mathcal{Z}[n,k]=1) = \frac{\psi^m(n,k)\xi(n,k)}{k}\\ \le\ & C(m)\,\frac{\psi^m(n_0,N)\xi(n)\xi(k)}{N}.\end{aligned} \tag{4.4}$$

Inserting (4.4) into (4.3) in combination with (4.2) yields

$$\begin{aligned}\mathbb{P}(\Delta\mathcal{Z}[n,j]=1,\Delta\mathcal{Z}[n,k]=0\ \forall k\in I_0|A_1)&\\ \ge \mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1)&\Big(1 - C(m)\,\frac{\psi^m(n_0,N)\xi(n)\xi(I_0)}{N}\Big),\end{aligned}$$

and using (4.1) yields the statement. □

Lemma 4.4 (Upper bound for conditional jump probabilities)**.** *For every $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ and $C > 0$, such that for $n_0 \le n \le N$ and $I_0, I_1 \subset \{n,\dots,N\}$ disjoint satisfying (4.1) and $\#I_1 \le m$,*

$$\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1,A_0) \le C\,\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1), \quad \textit{for all } j \in \{n,\dots,N-1\}.$$

Proof. This is a modification of [DM13, Lemma 2.12]. We have

$$\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_0,A_1) \le \frac{\mathbb{P}(\Delta\mathcal{Z}[n,j]=1|A_1)}{\mathbb{P}(A_0|A_1)},$$

so it suffices to bound $\mathbb{P}(A_0|A_1)$ uniformly from below. Since $\#I_1 \le m$, we get by Lemma 3.7,

$$\mathbb{P}(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0|\Delta\mathcal{Z}[n,j]=1\ \forall j\in I_1) \ge \mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0). \tag{4.5}$$

Denoting $i=\min I_0$, we obtain

$$\begin{aligned}\mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0\ \forall n\in I_0) =& \mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0\setminus\{i\}|\Delta\mathcal{Z}[n,i]=0)\\ &\times \mathbb{P}^m(\Delta\mathcal{Z}[n,i]=0)\\ \ge& \mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0\setminus\{i\})\mathbb{P}^m(\Delta\mathcal{Z}[n,i]=0),\end{aligned}$$

using Lemma 3.6. Iteration yields

$$\mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0) \ge \prod_{j\in I_0}\mathbb{P}^m(\Delta\mathcal{Z}[n,j]=0) = \prod_{j\in I_0}\big(1-\tfrac{1}{j}\mathbb{E}^m f(\mathcal{Z}[n,j])\big), \tag{4.6}$$

and inserting (4.6) into (4.5) yields

$$\mathbb{P}(\Delta\mathcal{Z}[n,j]=0\ \forall j\in I_0|\Delta\mathcal{Z}[n,j]=1\ \forall j\in I_1) \ge \prod_{j\in I_0}\big(1-\tfrac{1}{j}\mathbb{E}^m f(\mathcal{Z}[n,j])\big). \tag{4.7}$$

Choose n_0 large enough so that $n_0\le n\le j$ implies $\frac{1}{j}\mathbb{E}^m f(\mathcal{Z}[n,j])<1$. We hence find $c>1$ such that, $-\log(1-\frac{1}{j}\mathbb{E}^m f(\mathcal{Z}[n,j]))\le \frac{c}{j}\mathbb{E}^m f(\mathcal{Z}[n,j])$. Thus, taking the logarithm in (4.7), we bound, using (3.2), for some constant $C>0$,

$$-\log\mathbb{P}(A_0|A_1) \le \sum_{j\in I_0}\tfrac{c}{j}\mathbb{E}^m f(\mathcal{Z}[n,j]) = c\sum_{j\in I_0}\frac{\psi^m(n,j)\xi(n)}{j\xi(j)} \le C\frac{\xi(n)\psi^m(n,N)\xi(I_0)}{N},$$

and the last expression is uniformly bounded by (4.1). □

To choose $(\ell_k)_{k\ge1}$ suitably, we need to understand how the choice of cutoff points influences the growth of the score. To this end let $\mathcal{E}$ denote a configuration obtained after some stage of the exploration process, $V\subset \text{veiled}(\mathcal{E})$ and consider the random variable

$$S(V)=\xi(\{v\in V: v\leftrightarrow \text{active}(\mathcal{E})\})=\xi(\{v\in V:\exists a\in\text{active}(\mathcal{E}): v\leftrightarrow a\}).$$

The inclusion-exclusion principle yields the lower bound

$$S(V)\ge \sum_{v\in V}\xi(v)\sum_{a\in\text{active}(\mathcal{E})}\mathbb{1}\{a\leftrightarrow v\} - \sum_{v\in V}\xi(v)\sum_{\substack{a<b\\ a,b\in\text{active}(\mathcal{E})}}\mathbb{1}\{a\leftrightarrow v\leftrightarrow b\}. \tag{4.8}$$

To derive bounds on the probability that the second sum is positive, we define events

$$\begin{aligned}A_1(V,\mathcal{E})&=\{\exists v\in V; a<b; a,b\in\text{active}(\mathcal{E}):\Delta\mathcal{Z}[v,a-1]=\Delta\mathcal{Z}[v,b-1]=1\},\\ A_2(V,\mathcal{E})&=\{\exists v\in V; a<b; a,b\in\text{active}(\mathcal{E}):\Delta\mathcal{Z}[a,v-1]=\Delta\mathcal{Z}[b,v-1]=1\},\\ A_3(V,\mathcal{E})&=\{\exists v\in V; a<b; a,b\in\text{active}(\mathcal{E}):\Delta\mathcal{Z}[a,v-1]=\Delta\mathcal{Z}[v,b-1]=1\}.\end{aligned}$$

Proposition 4.5 (Collision probability)**.** *Let* $\mathbf{e}$ *be a proper configuration and* $V\subset\text{veiled}(\mathbf{e})$ *such that, for some fixed* $m\in\mathbb{N}$,

$$\xi(\min V)\xi(\text{active}(\mathbf{e})\cup\text{dead}(\mathbf{e}))\le C(m)\,\frac{N}{2\psi^m(n_0,N)}. \tag{4.9}$$

Then there is a constant $C>0$, depending only on f and m, such that

$$\mathbb{P}\Big(\bigcup_{i=1}^{3} A_i(V,\mathcal{E})\Big|\mathcal{E}=\mathbf{e}\Big)\le C\big(1\vee\log\tfrac{N}{\min V\wedge\min(\text{active}(\mathbf{e}))}\big)^{2\alpha+1}\frac{\xi(\text{active}(\mathbf{e}))^2-\xi^2(\text{active}(\mathbf{e}))}{N}.$$

Proof. Repeated use of the union bound yields

$$\mathbb{P}(A_1(V,\mathcal{E})|\mathcal{E}=\mathbf{e})\le\sum_{v\in V}\sum_{\substack{a,b\in\text{active}(\mathbf{e})\\a<b}}\mathbb{P}(\Delta\mathcal{Z}[v,a-1]=\mathcal{Z}[v,b-1]=1|\mathcal{E}=\mathbf{e})$$

Using Lemmas 3.7 and 4.4 and Proposition 3.2 we get, for some $B>0$,

$$\begin{aligned}\mathbb{P}(&\,\Delta\mathcal{Z}[v,a-1]=\Delta\mathcal{Z}[v,b-1]=1|\mathcal{E}=\mathbf{e})\\&=\mathbb{P}(\Delta\mathcal{Z}[v,b-1]=1|\mathcal{E}=\mathbf{e},\Delta\mathcal{Z}[v,a-1]=1)\,\mathbb{P}(\Delta\mathcal{Z}[v,a-1]=1|\mathcal{E}=\mathbf{e})\\&\le C_{4.4}^2\mathbb{P}^1(\Delta\mathcal{Z}[v,b-1]=1)\mathbb{P}(\Delta\mathcal{Z}[v,a-1]=1)\\&\le C_{4.4}^2C_{3.2}\frac{(\log\frac{b}{v}\vee 1)^\alpha(\log\frac{a}{v}\vee 1)^\alpha}{v\sqrt{ab}}\\&\le B\frac{\xi(a)\xi(b)}{vN}(\log\tfrac{b}{v}\vee 1)^\alpha(\log\tfrac{a}{v}\vee 1)^\alpha.\end{aligned}$$

Hence, with $v_0=\min V$, we get

$$\mathbb{P}(A_1(V,\mathcal{E})|\mathcal{E}=\mathbf{e})\le\frac{B}{N}(\log\tfrac{N}{v_0}\vee 1)^{2\alpha}\sum_{v\in V}\frac{1}{v}\sum_{\substack{a,b\in\text{active}(\mathbf{e})\\a<b}}\xi(a)\xi(b).$$

For $A_2(V,\mathcal{E})$ we need to take into account that, for $a\in\text{active}(\mathbf{e})$, $\mathcal{Z}[a,\cdot]$ may only be conditioned to have at most one jump, since $\mathbf{e}$ is proper. A calculation as above yields

$$\mathbb{P}(A_2(V,\mathcal{E})|\mathcal{E}=\mathbf{e})\le\frac{B'}{N}(\log\tfrac{N}{a_0}\vee 1)^{2\alpha}\sum_{v\in V}\frac{1}{v}\sum_{\substack{a,b\in\text{active}(\mathbf{e})\\a<b}}\xi(a)\xi(b),$$

for some $B'>0$ and $a_0=\min(\text{active}(\mathbf{e}))$. Analogously, we obtain

$$\mathbb{P}(A_3(V,\mathcal{E})|\mathcal{E}=\mathbf{e})\le\frac{B''}{N}(\log\tfrac{N}{a_0\wedge v_0}\vee 1)^{2\alpha}\sum_{v\in V}\frac{1}{v}\sum_{\substack{a,b\in\text{active}(\mathbf{e})\\a<b}}\xi(a)\xi(b),$$

for some $B''>0$. Setting $B'''=\max(B,B',B'')$ these three estimates together with the union bound yield

$$\mathbb{P}\Big(\bigcup_{i=1}^{3}A_i(V,\mathcal{E})\Big|\mathcal{E}=\mathbf{e}\Big)\le\frac{B'''}{N}(\log\tfrac{N}{a_0\wedge v_0}\vee 1)^{2\alpha}\big(\xi(\text{active}(\mathbf{e}))^2-\xi^2(\text{active}(\mathbf{e}))\big)\sum_{v\in V}\frac{1}{v},$$

which implies the claimed upper bound. □

This bound allows us to ignore the second sum of (4.8) outside a set of small probability. Decomposing the first sum of (4.8) according to the orientation of the occuring edges yields

$$\begin{aligned}\sum_{v\in V}\xi(v)\sum_{a\in\text{active}(\mathcal{E})}\mathbb{1}\{a\leftrightarrow v\}&=\sum_{v\in V}\sum_{a\in\text{active}(\mathcal{E})}\xi(v)\mathbb{1}\{v\leftarrow a\}+\sum_{a\in\text{active}(\mathcal{E})}\sum_{v\in V}\xi(v)\mathbb{1}\{a\leftarrow v\}\\&=:S^<(V)+S^>(V).\end{aligned}$$

Setting

$$X_v:=\sum_{a\in\text{active}(\mathcal{E})}\xi(v)\mathbb{1}\{v\leftarrow a\},\ v\in V\ \text{ and }\ Y_a:=\sum_{v\in V}\xi(v)\mathbb{1}\{a\leftarrow v\},\ a\in\text{active}(\mathcal{E}),$$

we note that due to the independence of indegree evolutions $S^{<}(V)$ and $S^{>}(V)$ are independent and both are sums of independent random variables. In order to apply Lemma 4.2 we determine moment bounds for $X_v, v \in V$, $Y_a, a \in \text{active}(\mathcal{E})$.

Proposition 4.6 (First and second moments of vertex scores)**.** *Let* $\mathbf{e}$ *be a proper configuration and* $V \subset \text{veiled}(\mathbf{e})$ *such that* (4.9) *is satisfied for some* $m \in \mathbb{N}$.

(i) There are constants $0 < c, C < \infty$ *depending only on* f *and* m*, such that for all* $v \in V$

$$\mathbb{E}[X_v|\mathcal{E} = \mathbf{e}] \;\geq\; c\frac{\xi(v)^2}{N} \sum_{\substack{a\in\text{active}(\mathbf{e}):\\ a>v}} \xi(a)(\log \tfrac{a}{v} \vee 1)^\alpha \tag{4.10}$$

and

$$\mathbb{E}[X_v^2|\mathcal{E} = \mathbf{e}] \;\leq C\xi(v)^2\Big(\sum_{\substack{a,b\in\text{active}(\mathbf{e}):\\ a,b>v,\ a\neq b}} \frac{(\log \frac{a}{v}\vee 1)^\alpha(\log \frac{b}{v}\vee 1)^\alpha}{v\sqrt{ab}} + \sum_{\substack{a\in\text{active}(\mathbf{e}):\\ a>v}} \frac{(\log \frac{a}{v}\vee 1)^\alpha}{\sqrt{va}}\Big). \tag{4.11}$$

(ii) There are constants $0 < c, C < \infty$ *depending only on* f *and* m*, such that for all* $a \in \text{active}(\mathbf{e})$

$$\mathbb{E}[Y_a|\mathcal{E} = \mathbf{e}] \;\geq\; c\frac{\xi(a)}{N} \sum_{\substack{v\in V:\\ v>a}} \xi(v)^2(\log \tfrac{v}{a} \vee 1)^\alpha \tag{4.12}$$

and

$$\mathbb{E}[Y_a^2|\mathcal{E} = \mathbf{e}] \;\leq C\Big(\sum_{\substack{v,w\in V:\\ v,w>a,\ v\neq w}} \xi(v)\xi(w)\frac{(\log \frac{v}{a}\vee 1)^\alpha(\log \frac{w}{a}\vee 1)^\alpha}{a\sqrt{vw}} + \sum_{\substack{v\in V:\\ v>a}} \xi(v)^2\frac{(\log \frac{v}{a}\vee 1)^\alpha}{\sqrt{va}}\Big). \tag{4.13}$$

Proof. As X_v is a constant multiple of a sum of indicators, its first conditional moment is

$$\begin{aligned}
\xi(v) \sum_{a\in\text{active}(\mathbf{e})} \mathbb{P}(\Delta\mathcal{Z}[v, a-1] = 1|\mathcal{E} = \mathbf{e}) &\geq \frac{1}{2}\xi(v) \sum_{a\in\text{active}(\mathbf{e})} \mathbb{P}(\Delta\mathcal{Z}[v, a-1] = 1) \\
&\geq \frac{c_{3.3}}{2}\xi(v) \sum_{\substack{a\in\text{active}(\mathbf{e}):\\ a>v}} \frac{(\log \frac{a}{v} \vee 1)^\alpha}{\sqrt{va}} \geq c\frac{\xi(v)^2}{N} \sum_{\substack{a\in\text{active}(\mathbf{e}):\\ a>v}} \xi(a)(\log \tfrac{a}{v} \vee 1)^\alpha,
\end{aligned}$$

where we have used Lemmas 3.6 and 4.3, Proposition 3.3, (3.2) and chosen some appropiate constant $c > 0$. A similar calculation for the second moment relies on Lemmas 3.6, 3.7 and 4.4 and Proposition 3.2 and reads

$$\begin{aligned}
&\xi(v)^2 \sum_{a,b\in\text{active}(\mathbf{e})} \mathbb{P}(\Delta\mathcal{Z}[v, a-1] = \Delta\mathcal{Z}[v, b-1] = 1|\mathcal{E} = \mathbf{e}) \\
&= \xi(v)^2 \sum_{a,b\in\text{active}(\mathcal{E})} \big(\mathbb{P}(\Delta\mathcal{Z}[v, a\vee b-1] = 1|\mathcal{E} = \mathbf{e}, \Delta\mathcal{Z}[v, a\wedge b-1] = 1) \\
&\qquad\qquad \times \mathbb{P}(\Delta\mathcal{Z}[v, a\wedge b-1] = 1|\mathcal{E} = \mathbf{e})\big) \\
&\leq C_{4.4}^2\xi(v)^2 \sum_{a,b\in\text{active}(\mathbf{e})} \mathbb{P}^1(\Delta\mathcal{Z}[v, a\vee b-1] = 1)^{\mathbb{1}\{a\neq b\}}\mathbb{P}(\Delta\mathcal{Z}[v, a\wedge b-1] = 1) \\
&\leq C_{4.4}^2C_{3.2}^2\xi(v)^2\Big(\sum_{\substack{a,b\in\text{active}(\mathbf{e}):\\ a,b>v,\ a\neq b}} \frac{(\log \frac{a}{v} \vee 1)^\alpha(\log \frac{b}{v} \vee 1)^\alpha}{v\sqrt{ab}} + \sum_{\substack{a\in\text{active}(\mathbf{e}):\\ a>v}} \frac{(\log \frac{a}{v} \vee 1)^\alpha}{\sqrt{va}}\Big),
\end{aligned}$$

This establishes (i). Turning to (ii) we obtain firstly, for some appropriately chosen $c > 0$,

$$\begin{aligned}\mathbb{E}[Y_a|\mathcal{E}=\mathbf{e}] &= \sum_{v\in V}\xi(v)\mathbb{P}(\Delta\mathcal{Z}[a,v-1]=1|\mathcal{E}=\mathbf{e})\\ &\geq \tfrac{1}{2}\sum_{v\in V}\xi(v)\mathbb{P}(\Delta\mathcal{Z}[a,v-1]=1)\geq c\,\xi(a)\sum_{v\in V:v>a}\frac{1}{v}(\log\tfrac{v}{a}\vee 1)^\alpha,\end{aligned}$$

where we have used Lemmas 3.6 and 4.3 for the first inequality and Proposition 3.3 and (3.2) for the second. Secondly, analogous to the second moment calculation for (i) we get

$$\begin{aligned}&\mathbb{E}[Y_a^2|\mathcal{E}=\mathbf{e}]\\ &= \sum_{v,w\in V}\xi(v)\xi(w)\mathbb{P}(\Delta\mathcal{Z}[a,v-1]=\Delta\mathcal{Z}[a,w-1]=1|\mathcal{E}=\mathbf{e})\\ &= \sum_{v,w\in V}\xi(v)\xi(w)\big(\mathbb{P}(\Delta\mathcal{Z}[a,v\vee w-1]=1|\mathcal{E}=\mathbf{e},\Delta\mathcal{Z}[a,v\wedge w-1]=1)\\ &\qquad\qquad\times\mathbb{P}(\Delta\mathcal{Z}[a,v\wedge w-1]=1|\mathcal{E}=\mathbf{e})\big)\\ &\leq C_{4.4}^2\sum_{v,w\in V}\xi(v)\xi(w)\mathbb{P}^2(\Delta\mathcal{Z}[a,v\vee w-1]=1)^{\mathbb{1}\{v\neq w\}}\mathbb{P}^1(\Delta\mathcal{Z}[a,v\wedge w-1]=1)\\ &\leq C_{4.4}^2C_{3.2}^2\Big(\sum_{\substack{v,w\in V:\\ v,w>a,\ v\neq w}}\xi(v)\xi(w)\frac{(\log\frac{v}{a}\vee 1)^\alpha(\log\frac{w}{a}\vee 1)^\alpha}{a\sqrt{vw}}+\sum_{\substack{v\in V\\ v>a}}\xi(v)^2\frac{(\log\frac{v}{a}\vee 1)^\alpha}{\sqrt{va}}\Big),\end{aligned}$$

for some $C<\infty$ and the claim follows. □

The lower bounds (4.10) and (4.12) now imply, that for $\mathbf{e}, V$ chosen as before

$$\begin{aligned}&\mathbb{E}[S^<(V)+S^>(V)|\mathcal{E}=\mathbf{e}]\\ &\geq \tfrac{c_{(4.10)}\wedge c_{(4.12)}}{N}\sum_{\substack{a\in\text{active}(\mathbf{e})\\ v\in V}}\mathbb{1}\{v<a\}\xi(v)^2\xi(a)\big(\log\tfrac{a}{v}\vee 1\big)^\alpha+\mathbb{1}\{v>a\}\xi(v)^2\xi(a)\big(\log\tfrac{v}{a}\vee 1\big)^\alpha\\ &\geq c\sum_{a\in\text{active}(\mathbf{e})}\xi(a)\sum_{v\in V}\frac{1}{v}\big(\log\tfrac{a\vee v}{a\wedge v}\vee 1\big)^\alpha,\end{aligned}\tag{4.14}$$

for some small $c>0$. The factor $\sum_{v\in V}\frac{1}{v}\big(\log\frac{a\vee v}{a\wedge v}\vee 1\big)^\alpha$ in the last sum is large as long as the set V is sufficiently dense in $[N]$. In fact, the following version of the pigeonhole principle holds, which is proved in Appendix A.

Lemma 4.7. *There are $\eta\in(0,1)$ and $c>0$ only depending on α such that for any choice of $A\subset\{\lceil 2\mathrm{e}^2\rceil,\dots,N\}$ and $v_0<\frac{\min A}{\mathrm{e}^2}\wedge\eta N$ satisfying*

$$\big(\log\tfrac{N}{\min A}\vee 1\big)^\alpha\xi^2(A)\leq\frac{c}{2}N\big(\log\tfrac{N}{v_0}\big)^{\alpha+1},\tag{4.15}$$

we have, for $V=\{v_0,\dots,N\}\setminus A$ and any $a\in A$,

$$\sum_{v\in V}\frac{1}{v}\big(\log\tfrac{a\vee v}{a\wedge v}\vee 1\big)^\alpha\geq\frac{c}{2}\big(\log\tfrac{N}{v_0}\big)^{\alpha+1},\tag{4.16}$$

if N is suffciently large.

We summarise our observations in the following lemma.

Lemma 4.8 (Concentration of score)**.** *Let* $\mathbf{e}$ *be a proper configuration with* $a_0 = \min(\mathrm{active}(\mathbf{e}))$ *and let* $v_0 < \frac{a_0}{\mathrm{e}^2} \wedge \eta_{4.7} N$ *such that* $V = \{v_0, \dots, N\} \cap \mathrm{veiled}(\mathbf{e})$ *satisfies* (4.9) *for* $m = 2$ *and furthermore*

$$\xi^2(\mathrm{active}(\mathbf{e})) \le \frac{c_{4.7}}{2} N \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} \big(\log \tfrac{N}{a_0} \vee 1\big)^{-\alpha}. \tag{4.17}$$

Then there exists a constant $c > 0$ *such that, for all* $\beta \in (0,1)$*,*

$$\begin{aligned}\mathbb{P}\big(S(V) &\le (1-\beta) \tfrac{c_{(4.14)} c_{4.7}}{2} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} \xi(\mathrm{active}(\mathbf{e})) \big| \mathcal{E} = \mathbf{e}\big) \\ &\le \exp\big(-\beta^2 c \min\big\{ \tfrac{\xi(\mathrm{active}(\mathbf{e}))^2}{\xi^2(\mathrm{active}(\mathbf{e}))}, \tfrac{\xi(\mathrm{active}(\mathbf{e}))(\log(N v_0^{-1}))^{\alpha+2}}{\xi(v_0)}, v_0 (\log(N v_0^{-1}))^2 \big\}\big) \\ &\quad + C_{4.5} \big(\log \tfrac{N}{v_0} \vee 1\big)^{2\alpha+1} \frac{\xi(\mathrm{active}(\mathbf{e}))^2 - \xi^2(\mathrm{active}(\mathbf{e}))}{N}.\end{aligned}$$

Proof. On the complement of the event $A_1(V,\mathcal{E}) \cup A_2(V,\mathcal{E}) \cup A_3(V,\mathcal{E})$ we choose

$$\lambda = \beta \tfrac{c_{(4.14)} c_{4.7}}{2} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} \xi(\mathrm{active}(\mathbf{e}))$$

and apply Lemma 4.2. The exponential in the conclusion of the lemma is then obtained by applying the estimates (4.11), (4.13), (4.14) and Lemma 4.7. The second term bounds the probability of the occurence of $A_1(V,\mathcal{E}) \cup A_2(V,\mathcal{E}) \cup A_3(V,\mathcal{E})$ using Proposition 4.5. □

As a consequence of Lemma 4.8 we bound the growth of the score from below. To this end define, for given $s_0 > 0$ and $\delta_0 \in (0, \frac{1}{2})$,

$$\ell_k := \max\Big\{ n \in [N] : \sqrt{\frac{N}{n}} \ge \frac{s_0 \delta_0}{1 \vee \log k} \prod_{i=1}^{k-1} \tfrac{c_{(4.14)} c_{4.7}}{4} \big(\log \tfrac{N}{\ell_i}\big)^{\alpha+1} \Big\} \text{ for all } k \ge 1. \tag{4.18}$$

If the maximum in the above definition is taken over the empty set, we let $\ell_k = 1$. Denoting by $k^*(N)$ the first index for which $\ell_{k+1} = \ell_k$, we check that $(\ell_k)_{k \ge 1}$ satisfies the following decay condition.

Lemma 4.9. *For any* $\alpha \ge 0, \delta \in (0, 2\alpha+2)$ *let*

$$k_0(\delta, \alpha) = \min\{k \ge 3 : \delta \log k \ge (2\alpha + 2 - \delta) k \log(1 + \tfrac{1}{k}) + 1\}$$

then

$$\ell_k \le N \mathrm{e}^{-(2\alpha+2-\delta)(k-k_0)\log k} \quad \textit{for all } k_0 \le k < k^*(N).$$

Also, there is a constant $c > 0$ *depending only on* s_0 *such that*

$$\ell_k \ge c N \mathrm{e}^{-(4\alpha+5)k(1 \vee \log k)}, \quad \textit{for all } k.$$

A verification of Lemma 4.9 is provided in Appendix A. We conclude this section with the central result on the growth of the score in the truncated exploration.

Proposition 4.10 (Score growth)**.** *Let* $\varepsilon, \eta > 0$ *and set*

$$K = \Big\lceil \Big(\frac{1}{2\alpha+2} + \eta\Big) \frac{\log N}{\log \log N} \Big\rceil.$$

Then there are $s_0(\varepsilon) > 0, \delta_0(\varepsilon) \in (0, \frac{1}{2})$ *and* $N_0(\varepsilon, \eta)$ *such that*

$$\mathbb{P}\Big(\xi(\mathrm{active}(\mathcal{E}_K) \cup \mathrm{dead}(\mathcal{E}_K)) \le \frac{\sqrt{N}}{(\log N)^{\alpha+1}}\Big) \le \varepsilon, \quad \textit{for all } N \ge N_0,$$

where $(\mathcal{E}_k)_{k \ge 0}$ *is the exploration in* $\mathcal{G}_N$ *with truncation* $(\ell_k)_{k \ge 1}$ *as in* (4.18) *that is started in a proper configuration* $\mathcal{E}_0$ *satisfying* $\frac{\xi(\mathrm{active}(\mathcal{E}_0))}{\xi(\min(\mathrm{active}(\mathcal{E}_0)))} \ge s_0$.

Proof. We first note that, for fixed $\eta > 0$, $k^* = k^*(N) < K$ for all sufficiently large N by Lemma 4.9. We wish to iteratively apply Lemma 4.8 until $k \le k^*(N)$ is so large that Lemma 4.9 allows to establish the lower bound for $\xi(\text{active}(\mathcal{E}_K) \cup \text{dead}(\mathcal{E}_K))$. To this end let, for $k \ge 0$,

$$S_k := \xi(\text{active}(\mathcal{E}_k)), \quad H_k := \xi(\text{active}(\mathcal{E}_k) \cup \text{dead}(\mathcal{E}_k)), \text{ and } \quad a_k := \min(\text{active}(\mathcal{E}_k)).$$

Let also $K_0 = K_0(N)$ denote the first stage k of the exploration during which

$$H_k > \frac{\sqrt{N}}{(\log N)^{\alpha+1}}$$

is first satisfied. The proof is complete once we have verified the following three claims:

(i) For given ε, δ_0 we may choose $s_0 > 0$ such that, for all sufficiently large N, the configuration $\mathcal{E}_0$ and ℓ_1 satisfy the conditions of Lemma 4.8 unless $K_0 = 0$. Additionally, with probability exceeding $1 - \gamma_1$, for $\gamma_1 := \frac{6\varepsilon}{2\pi^2}$ we have, for all sufficiently large N,
$$S_1 > \tfrac{c_{(4.14)}c_{4.7}}{4}\big(\log \tfrac{N}{\ell_1}\big)^{\alpha+1} S_0.$$

(ii) Conditional on $k < K_0 \wedge k^*$ and $\mathcal{E}_j, 1 \le j \le k$, satisfying
$$S_j > \tfrac{c_{(4.14)}c_{4.7}}{4}\big(\log \tfrac{N}{\ell_j}\big)^{\alpha+1} S_{j-1},$$
the configuration $\mathcal{E}_k$ and ℓ_{k+1} satisfy the conditions of Lemma 4.8. Consequently, we can find $\gamma_k > 0$, such that with conditional probability exceeding $1 - \gamma_k$ we have
$$S_{k+1} > \tfrac{c_{(4.14)}c_{4.7}}{4}\big(\log \tfrac{N}{\ell_{k+1}}\big)^{\alpha+1} S_k,$$
and thus $k^* \ge K_0 \ge k+1$.

(iii) $\delta_0 = \delta_0(\varepsilon)$ may be fixed in such a way that $(\gamma_k)_{k\ge1}$ from (i) and (ii) satisfies $\sum_{k=1}^{L} \gamma_k < \varepsilon$ as $N \to \infty$ for any $L = O(\log N (\log\log N)^{-1})$.

Proof of (i): If $K_0 = 0$, then there is nothing to show. Let $K_0 > 0$. Given δ_0, the condition $\ell_1 < \frac{a_0}{e^2}$ is satisfied by choosing s_0 sufficiently large. The conditions $\ell_1 < \eta_{4.7}N$, (4.9) and (4.17) are now implicit in the assumption $K_0 > 0$, for all sufficiently large N. Application of Lemma 4.8 with $\beta = \frac{1}{2}$ yields

$$\mathbb{P}\Big(S_1 \le \tfrac{c_{(4.14)}c_{4.7}}{4}\big(\log \tfrac{N}{\ell_1}\big)^{\alpha+1}\xi(\text{active}(\mathcal{E}_0))\Big) \le e^{-\frac{c_{4.8}}{4}s_0} + o(1),$$

as $N \to \infty$, thus, after possibly increasing s_0 again, (i) holds.

Proof of (ii): Assume that $k^* > K_0 > k$ and note that this implies $\xi(\ell_{k+1}) \le \sqrt{N}$. By definition of the exploration we have $a_k \ge \ell_k$. By Lemma 4.9 and the definition of $(\ell_k)_{k\ge1}$, the network size N can be chosen so large that $(\ell_k)_{k\ge1}$ decays faster than $(e^{-2k})_{k\ge1}$ for all $k < k^*(N)$. In particular $\ell_{k+1} < \frac{a_k}{e^2}$ holds. As in the proof of (i), $K_0 > k$ implies that (4.17) is satisfied and also, using $\xi(\ell_{k+1}) \le \sqrt{N}$, (4.9) must hold. Hence we may again apply Lemma 4.8 with $\beta = \frac{1}{2}$ to obtain that

$$\begin{aligned}
&\mathbb{P}\Big(S_{k+1} \le \tfrac{c_{(4.14)}c_{4.7}}{4}\big(\log \tfrac{N}{\ell_{k+1}}\big)^{\alpha+1} S_k\Big)\\
&\le \exp\Big(-\tfrac{c_{4.8}}{4}\min\Big\{\tfrac{S_k^2}{\xi^2(\text{active}(\mathcal{E}_k))}, \tfrac{S_k(\log(N\ell_{k+1}^{-1}))^{\alpha+2}}{\xi(\ell_{k+1})}, \ell_{k+1}(\log(N\ell_{k+1}^{-1}))^2\Big\}\Big)\\
&\qquad\qquad\qquad + C_{4.5}\big(\log \tfrac{N}{\ell_{k+1}} \vee 1\big)^{2\alpha+1}\frac{S_k^2}{N}\\
&=: \Delta_k + \Gamma_k.
\end{aligned} \tag{4.19}$$

The conclusion $K_0 \geq k^*$ follows, as $\frac{S_{k+1}}{S_k} \geq \frac{\xi(\ell_{k+1})}{\xi(\ell_k)}$.

Proof of (iii): It remains to bound the random sums $\Delta_k + \Gamma_k, k \leq K_0$, appearing in (4.19) by some deterministic sequence γ_k with the desired summability property. We start with Γ_k. Since $S_k < H_{K_0}$, we get

$$\Gamma_k \leq C_{4.5}(\log N)^{2\alpha+1}\frac{S_k^2}{N} \leq \frac{C_{4.5}}{\log N},$$

thus $\sum_{k=1}^{L}\Gamma_k = O((\log\log N)^{-1})$ for all $L = O(\log N(\log\log N)^{-1})$. To bound Δ_k, we analyse the three terms under maximisation separately. Since $x \mapsto x(\log\frac{N}{x})^2$ is strictly increasing on $[1, \frac{N}{\mathrm{e}^2}]$, the deterministic rightmost term satisfies

$$\ell_{k+1}\big(\log\tfrac{N}{\ell_{k+1}}\big)^2 \geq (\log N)^2.$$

By definition of $(\ell_k)_{k\geq 1}$ and (3.2) there is some constant c, independent of k, N and ε, such that $\xi(\ell_{k+1}) \leq c(\log\frac{N}{\ell_k})^{\alpha+1}\xi(\ell_k)$ and thus

$$\frac{S_k(\log(\frac{N}{\ell_{k+1}}))^{\alpha+2}}{\xi(\ell_{k+1})} \geq \frac{S_k(\log(\frac{N}{\ell_{k+1}}))^{\alpha+2}}{c\big(\log\frac{N}{\ell_k}\big)^{\alpha+1}\xi(\ell_k)} \geq \frac{S_k}{c\xi(\ell_k)}.$$

Since

$$\sum_{u\in\mathrm{active}(\mathcal{E}_k)}\xi^2(u) \leq \xi(a_k)S_k \leq \xi(\ell_k)S_k,$$

we also have

$$\frac{S_k^2}{\xi^2(\mathrm{active}(\mathcal{E}_k))} \geq \frac{S_k}{\xi(\ell_k)}.$$

On the conditioning event of (ii), we have

$$S_k \geq s_0\prod_{i=1}^{k}\frac{c_{(4.14)}c_{4.7}}{4}\big(\log\tfrac{N}{\ell_i}\big)^{\alpha+1}$$

and thus

$$\frac{S_k}{\xi(\ell_k)} \geq (1\vee\log k)\frac{c'}{\delta_0}$$

for some constant $c' > 0$. Combining all estimates, we obtain, for some $c'' > 0$,

$$\Delta_k \leq \mathrm{e}^{-c''\left(\frac{(1\vee\log k)}{\delta_0}\wedge(\log N)^2\right)}.$$

This implies that by choosing δ_0 small enough we may obtain $\Delta_k \leq \frac{6\varepsilon}{2\pi^2k^2} \vee N^{-c''\log N}$ and the last claim is proved. $\square$

4.3. **Connectivity of high degree vertices.** We now provide a connectivity result for those vertices in $\mathcal{G}_N$ which have a very high degree. The instrument used for this is a sprinkling-type argument based on an analogue of *t-connectors* as introduced in [DHH10].

Let $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$ and $u \neq v \in [N]$ be vertices in $\mathcal{G}_N$. Then

$$n \in \{N+1, \dots, \lfloor(1+\varepsilon)N\rfloor\}$$

is called an *ε-connector* for u and v if both $n \to u$ and $n \to v$. We write

$$\{u \overset{\varepsilon}{\leftrightarrow} v\} = \{\exists\, n \in \{N+1, \dots, (1+\varepsilon)N\} \;:\; n \text{ is an } \varepsilon\text{-connector for } u \text{ and } v\},$$

and extend the notation also to sets, i.e. if $A, B \subset [N]$ are disjoint, then $\{A \overset{\varepsilon}{\leftrightarrow} B\} = \{\exists\, (a,b) \in A \times B \,:\, a \overset{\varepsilon}{\leftrightarrow} b\}$. Clearly $u \overset{\varepsilon}{\leftrightarrow} v$ implies $d_{\lceil (1+\varepsilon)N\rceil}(u,v) \le 2$. Furthermore, given $\mathcal{G}_N$ and $a, b \in [N]$, for all potential ε-connectors $N+1 \le n \le (1+\varepsilon)N$, we have

$$\mathbb{P}\big(n \text{ is an } \varepsilon\text{-connector for } a \text{ and } b \big|\, \mathcal{G}_N\big) \ge \frac{f(\mathcal{Z}[a,N])f(\mathcal{Z}[b,N])}{((1+\varepsilon)N)^2}, \tag{4.20}$$

which follows from the model definition and the fact that the evolutions $\mathcal{Z}[a,\cdot], \mathcal{Z}[b,\cdot]$ are non-decreasing. Note that the bound holds independently for each ε-connector.

We are interested in bounds for the diameter of certain subsets of the oldest part of the graph arising from two-step connections using connectors. This graph will be coupled to an Erdős-Rényi graph with sufficiently large edge density. Let

$$\sigma := \sigma(\varepsilon) := \frac{3 + \inf_k \xi(k, \lceil (1+\varepsilon)k\rceil)}{4} - 1 \in (0,1). \tag{4.21}$$

Let $M = \lfloor (\log N)^R \rfloor$ for some fixed $R > \alpha$. We call $v \in [M]$ a *core vertex* of $\mathcal{G}_{\lceil (1+\varepsilon)N\rceil}$ if

$$f(\mathcal{Z}[v,N]) \ge \tfrac{1}{2}\mathbb{E} f(\mathcal{Z}[M,N]) \quad \text{and } f(\mathcal{Z}[v,\lceil (1+\varepsilon)N\rceil]) - f(\mathcal{Z}[v,N]) \ge \sigma f(\mathcal{Z}[M,N]).$$

Let C_N be the subset of $[M]$ containing all core vertices, and let

$$E_N = \{(v,w) \in C_N \times C_N \,:\, v \overset{\varepsilon}{\leftrightarrow} w\}$$

be the edge set of the random graph core_N, the *core* of $\mathcal{G}_{\lceil (1+\varepsilon)N\rceil}$. We now establish a lower bound on the edge density of core_N to allow coupling to a suitable Erdős-Rényi graph.

Lemma 4.11 (Size of core_N). *Fix $\varepsilon > 0$ and $M = \lfloor (\log N)^R \rfloor$ for some $R > \alpha$.*

(i) There exists $q(f,\varepsilon) > 0$ such that

$$\mathbb{P}(v \in C_N) \ge q(f,\varepsilon) \quad \textit{for all } v \in [M] \textit{ independently.}$$

(ii) With high probability $\#C_N \ge M\frac{q(f,\varepsilon)}{2}$, as $N \to \infty$.

Proof. (i) Note that the conditions on a vertex v to be a core vertex are conditions on the growth of the degree evolution $(\mathcal{Z}[v,\cdot])$ only. Independence of the indicators $\mathbb{1}\{v \in C_N\}$, $v \in [N]$, thus follows from the independence of degree evolutions. By monotonicity, it is sufficient to verify the conditions only for $v = M$. Since $f(k) \ge \frac{k}{2}$, for all k, we only need to show the existence of a small positive constant q, such that with probabilty exceeding q, we have

$$f(\mathcal{Z}[M,N]) \ge \tfrac{1}{2}\mathbb{E} f(\mathcal{Z}[M,N]) \tag{4.22}$$

and

$$f(\mathcal{Z}[M,\lceil (1+\varepsilon)N\rceil]) \ge (1+\sigma) f(\mathcal{Z}[M,N]), \tag{4.23}$$

for all sufficiently large N. To show (4.22), note that the Paley-Zygmund inequality implies

$$\mathbb{P}\big(f(\mathcal{Z}[M,N]) \ge \tfrac{1}{2}\mathbb{E} f(\mathcal{Z}[M,N])\big) \ge \frac{(\mathbb{E} f(\mathcal{Z}[M,N]))^2}{4\mathbb{E} f(\mathcal{Z}[M,N])^2} =: p(M,N).$$

The moment bounds of Proposition 3.2 now entail that $p(M,N) > p_1 > 0$ for some small p_1 and all sufficiently large N. Let $\mathcal{Z}[n,n] = \bar{\mathcal{Z}}[n,n] = k$ and denote by $\bar{\mathcal{Z}}$ the degree evolution associated with the linearised attachment rule satisfying $\bar f(l) = f(k) + \frac{1}{2}(l-k) \le f(l)$, $l \ge k$. As f is concave, $(\mathcal{Z}[n,s])_{s\ge n}$ stochastically dominates $(\bar{\mathcal{Z}}[n,s])_{s\ge n}$. Thus, for some $p_2 > 0$,

$$\begin{aligned}\mathbb{P}\big(f(\mathcal{Z}[M,&\lceil(1+\varepsilon)N\rceil]) \ge (1+\sigma)f(\mathcal{Z}[M,N]) | \mathcal{Z}[M,N] = k\big)\\ &\ge \mathbb{P}^k\big(\bar f(\bar{\mathcal{Z}}[N,\lceil(1+\varepsilon)N\rceil]) \ge (1+\sigma)f(k)\big)\\ &\ge \Big(1 - \frac{1+\sigma}{\inf_k \xi(k,\lceil(1+\varepsilon)k\rceil)}\Big)^2 \frac{(\mathbb{E}\bar f(\bar{\mathcal{Z}}[N,\lceil(1+\varepsilon)N\rceil]))^2}{\bar f(\mathbb{E}\bar{\mathcal{Z}}[N,\lceil(1+\varepsilon)N\rceil])^2} > p_2,\end{aligned}$$

where we have used the Paley-Zygmund inequality and Lemma 3.1 for the last two steps. Thus (4.23) holds with probability at least p_2, independently of the value of $\mathcal{Z}[M,N]$. Since $(\mathcal{Z}[M,s])_{s\ge M}$ is a Markov process, the increment $f(\mathcal{Z}[M,\lceil(1+\varepsilon)N\rceil]) - f(\mathcal{Z}[M,N])$ can only depend on the past via $\mathcal{Z}[M,N]$. It follows that (4.22) and (4.23) are jointly satisfied with probability at least $q = p_1 p_2$.

(ii) Note that by (i) all $v \in [M]$ have probability at least $q = q(f,\varepsilon)$ to be in the core and the degree evolutions are independent. Thus $\#C_N$ dominates the sum $S_M = \sum_{i=1}^M X_i$, where $X_1, X_2, \dots$ are i.i.d. Bernoulli with parameter q. Chernoff's inequality now yields $\mathbb{P}(\#C_N < M\frac{q}{2}) \le \mathbb{P}(S_M < M\frac{q}{2}) \le \exp(-\frac{q}{8}M)$, which converges to zero as $N \to \infty$. □

Next we show that the core is of bounded diameter.

Proposition 4.12 (Diameter of the core)**.** *For any $\varepsilon > 0$, core$_N \subset \mathcal{G}_{\lceil(1+\varepsilon)N\rceil}$ has bounded diameter with high probability, as $N \to \infty$.*

Proof. For $v, w \in C_N$ we first provide an estimate for the connection probability which is *independent* of the occupation status of the other edges in core$_N$. Recall that $\psi(n,N) = \psi^0(n,N) \to \infty$ if $\frac{N}{n} \to \infty$, by Proposition 3.3. Fix $v \in C_N$. Since $f(\mathcal{Z}[v,\lceil(1+\varepsilon)N\rceil]) \ge (1+\sigma)f(\mathcal{Z}[v,N])$, we know that by concavity of f,

$$\mathcal{Z}[v,\lceil(1+\varepsilon)N\rceil] - \mathcal{Z}[v,N] \ge \sigma\mathcal{Z}[v,N] \ge \frac{\sigma}{2}\mathbb{E}f(\mathcal{Z}[v,N]).$$

Let $Z(v) = \{N < j \le \lceil(1+\varepsilon)N\rceil \,:\, \Delta\mathcal{Z}[v,j] = 1\}$. Since $v \le M$, the number $\#Z(v)$ of potential ε-connectors to v is thus bounded below by

$$\frac{\sigma}{4}\mathbb{E}\mathcal{Z}[M,N] \ge \eta\sqrt{\tfrac{N}{M}},$$

for some $\eta = \eta(\varepsilon) > 0$. Given $\mathcal{G}_N$ and $v \in C_N$, the probability that there is no ε-connector connecting v and w can therefore be bounded as follows,

$$\mathbb{P}\big((v,w) \notin E_N\big) = \prod_{j\in Z(v)} \big(1 - \tfrac{f(\mathcal{Z}[w,j])}{j}\big) \le \big(1 - \tfrac{f(\mathcal{Z}[w,N])}{\lceil(1+\varepsilon)N\rceil}\big)^{\eta\sqrt{\frac{N}{M}}}.$$

Since $w \in C_N$, we can find $c = c(\varepsilon) > 0$, such that $f(\mathcal{Z}[w,N]) \ge c\psi(M,N)(\frac{N}{M})^{\frac{1}{2}}$, and thus

$$\mathbb{P}\big((v,w) \notin E_N\big) \le \Big(1 - \tfrac{c\psi(M,N)\sqrt{\frac{N}{M}}}{\lceil(1+\varepsilon)N\rceil}\Big)^{\eta\sqrt{\frac{N}{M}}} \le \exp\Big(-\tfrac{\eta c\psi(M,N)\frac{N}{M}}{\lceil(1+\varepsilon)N\rceil}\Big).$$

The argument of the exponential is close to 0, for such arguments x we have that $\mathrm{e}^{-x} \le 1 - \frac{x}{2}$. For sufficiently large N, we thus obtain

$$\mathbb{P}\big((v,w) \notin E_N\big) \le 1 - \frac{c\eta\psi(M,N)\frac{N}{M}}{2\lceil(1+\varepsilon)N\rceil}.$$

We have therefore shown, that for some small $\delta = \delta(\varepsilon,\alpha) > 0$, and N sufficiently large,

$$\mathbb{P}\big((v,w) \in E_N\big) \ge \delta\frac{\psi(M,N)}{M}. \tag{4.24}$$

Note that, given $\mathcal{G}_N$ and C_N, this holds independently for all pairs $v, w \in C_N$. We can thus dominate core_N by an Erdős-Rényi graph $\mathcal{G}(n,p)$ of random size $n = \#C_N$ and connection probability $p = \delta\psi(M,N)M^{-1}$. The classical diameter result for $\mathcal{G}(n,p)$, see e.g. [Bol01, Corollary 10.12], now implies that the diameter of the core is bounded by $\lceil \frac{R}{\alpha} \rceil + 1$. □

4.4. **Proof of Theorem 1.** It remains to prove the upper bound by combining the results about the first two phases of the explorations of *two* independently chosen vertices, and join the connected components uncovered during these explorations to core_N.

Proof of Theorem 1. Fix $\varepsilon > 0$. We start local explorations in the uniformly chosen vertices $U, V \in \mathcal{C}_N$. By Proposition 4.1 there exists $k_0(\varepsilon)$ such that with probability exceeding $1 - \frac{\varepsilon}{4}$ we reach after at most $k \leq k_0(\varepsilon)$ exploration steps in both explorations active sets $A \subset \mathfrak{T}_k$ satisfying $\xi(A) \geq s_0 \xi(\min A)$ with $s_0 = s_0\big(\frac{\varepsilon}{8}\big)$, as defined in Proposition 4.10. We start the main phase of the two explorations with initial configurations in which the sets A represent the active vertices, and possible other active vertices are veiled and connecting edges removed. Observe that this modification can only increase the distance of U and V. We denote the explored parts of the network at this stage by $\mathcal{E}_0^{(1)}, \mathcal{E}_0^{(2)}$ and henceforth only look at the scores of the two explorations. To keep the explorations sufficiently independent, we slightly modify the algorithm: In every exploration step, the exploration around U only checks for connections to the right for *odd* vertices, whereas the exploration around V only checks for connections to the right for *even* vertices. It is easily seen that this only changes the constant in the lower bound of Lemma 4.7.

We know by Proposition 4.10 that, if N is sufficiently large, then both explorations if viewed on their own end with probability exceeding $1 - \frac{\varepsilon}{4}$ after $K_0^{(i)} \leq (\frac{1}{2\alpha+2} + \frac{\eta}{2}) \frac{\log N}{\log\log N}$ steps, for any choice of $\eta > 0$. By this point

$$H_K^{(i)} \geq \frac{\sqrt{N}}{(\log N)^{1+\alpha}}, \tag{4.25}$$

where $H_K^{(i)} = \xi(\text{active}(\mathcal{E}_{K_0^{(i)}}) \cup \text{dead}(\mathcal{E}^{(i)}_{K_0^{(i)}}))$. Also note that, by choice of $(\ell_k)_{k \geq 1}$,

$$H_K^{(i)} \geq d\, \frac{\log N}{\log\log N}\, \xi(\ell_K), \tag{4.26}$$

for some small $d > 0$. We may assume without losing generality that $K_0^{(1)} < K_0^{(2)}$. After stage $K_0^{(1)}$, we cannot apply exactly the same reasoning for the second exploration as in Proposition 4.10, since the total score of both configurations is too high. However, the lower bound given in Lemma 4.3 can still be applied in each exploration step, since the set I_0 of non-jump times featured in this lemma consists only of odd vertices and is therefore disjoint of the sets of non-jump times used in the other exploration which may have exceeded the score bounds. The restriction on the set of jump-times I_1 clearly plays no role – if we encounter an additional jump due to a connection to the first exploration, then the procedure can be stopped and a shortest path connecting U and V is found.

As a consequence, we deduce that with high probability, U and V are either found to be connected before stage $K_0^{(2)}$ or their respective explorations have reached a score of at least $\sqrt{N}(\log N)^{-(\alpha+1)}$. Since $S_{K_0} \geq \xi(\ell_{K_0+1})$ by construction, we have collected no information about the degree evolutions of vertices in $[M]$, where $M = \lfloor \frac{1}{2}(\log N)^{2\alpha+2} \rfloor$, during the main phase of the exploration. Therefore we can apply Lemma 4.11 and Proposition 4.12 to deduce that, for sufficiently large N, with probability exceeding $1 - \frac{\varepsilon}{4}$, the subgraph core_N is of bounded diameter D and contains at least rM vertices, for some $r = r(\varepsilon) > 0$.

Denote the sets of active and dead vertices of $\mathcal{E}^{(i)}_{K_0^{(i)}}$ by $V(i)$. It remains to show that

$$\mathbb{P}\big(V(i) \overset{\varepsilon}{\leftrightarrow} \mathrm{core}_N, i \in \{1,2\}\big) \geq 1 - \frac{\varepsilon}{4},$$

if N is sufficiently large. Given $\mathcal{G}_N, \mathrm{core}_N$, let $L = \{j \in \{N+1, \ldots, \lceil (1+\varepsilon)N\rceil\} : j \leftrightarrow \mathrm{core}_N\}$. In the last paragraph, we have already established that with high probability core_N contains at least rM vertices. It is now straightforward to deduce via an appropriate coupling to Bernoulli random variables that $\#L \geq \lfloor qM\psi(M,N)\xi(M)\rfloor =: l_0$ with probability at least $1 - \frac{\varepsilon}{12}$, where $q = q(\varepsilon) > 0$ is some small constant.

Each $j \in L$ has an independent probability of at least $1 - f(\mathcal{Z}[v,N])/(1+\varepsilon)N$ to connect to $v \in V(i)$, thus the probability that it does not connect to any $v \in V(i)$ is bounded above by $\exp\big(-((1+\varepsilon)N)^{-1}\sum_{v\in V(i)} f(\mathcal{Z}[v,N])\big)$. Since this holds independently for all $j \in L$, we obtain by (4.25) and the definition of l_0,

$$\begin{aligned}\mathbb{1}\{\#L \geq l_0\}\mathbb{1}\big\{ \textstyle\sum_{v\in V(i)} f(\mathcal{Z}[v,N]) \geq \nu\xi(V(i))\big\}\mathbb{P}\big(\{V(i) \overset{\varepsilon}{\leftrightarrow} \mathrm{core}_N\}^{\mathrm{c}} \,\big|\, \mathcal{G}_N, \mathrm{core}_N)\big) \\ \leq \exp\big(-\tfrac{l_0}{\nu(1+\varepsilon)N} H_K^{(i)}\big) \leq \tfrac{\varepsilon}{24},\end{aligned} \tag{4.27}$$

for all sufficiently large N and fixed $\nu \in (0,1)$. It remains to fix $\nu > 0$ and bound

$$\mathbb{P}\Big(\sum_{v\in V(i)} f(\mathcal{Z}[v,N]) < \nu\xi(V(i))\Big) = \mathbb{P}\Big(\sum_{v\in V(i)} f(\mathcal{Z}[v,N]) < \nu H_K^{(i)}\Big).$$

The proof of Proposition 4.10 shows that $(S_k)_{k=1}^{K_0}$ grows superexponentially, thus for every $\mu > 0$, there is $\nu > 0$ such that

$$\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} f(\mathcal{Z}[v,N]) \geq \mu S_{K_0} \Rightarrow \sum_{v\in V(i)} f(\mathcal{Z}[v,N]) \geq \nu H_{K_0}.$$

Note that, for $v \in \mathrm{active}(\mathcal{E}_{K_0})$, replacing the attachment rule f by the linearised attachment rule $\bar{f}(k) = f(0) + \frac{k}{2}$ does not change the values $\xi(v)$ and only diminishes the sum on the left. For the rest of the argument we may therefore assume that $f = \bar{f}$ in the evolutions $\{\mathcal{Z}[v,\cdot], v \in \mathrm{active}(\mathcal{E}_{K_0})\}$. During the final exploration stage K_0, the evolution $\mathcal{Z}[v,i]_{i=1}^N$ of an active vertex v is only conditioned on a set I_0 of non-jumps which still fullfills the conditions of Lemma 4.3. This implies that, for some small $s > 0$, we have $\mathbb{E}[f(\mathcal{Z}[v,N])|\mathcal{E}_{K_0}] \geq s\xi(v)$, and thus

$$\mathbb{E}\Big[\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} f(\mathcal{Z}[v,N])\Big|\mathcal{E}_{K_0}\Big] \geq sS_{K_0}.$$

The random variables under summation on the left are independent. Choosing $\mu = \mu(s)$ small enough we thus find, by Lemma 4.2,

$$\mathbb{P}\Big(\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} f(\mathcal{Z}[v,N]) < \mu S_{K_0} \,\Big|\, \mathcal{E}_{K_0}\Big) \leq \exp\Big(-\delta\frac{S_{K_0}^2}{\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} \mathbb{E}[f(\mathcal{Z}[v,N])^2|\mathcal{E}_{K_0}]}\Big),$$

for some $\delta = \delta(\mu) > 0$. Taking into account the linearisation of f, and Proposition 3.2, we obtain $\mathbb{E}[f(\mathcal{Z}[v,N])^2|\mathcal{E}_{K_0}] \leq C_{3.2}\mathbb{E}f(\mathcal{Z}[v,N])^2 \leq C\xi(v)^2$, for some constant $C > 0$. Hence

$$\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} \mathbb{E}\big[f(\mathcal{Z}[v,N])^2 \,\big|\, \mathcal{E}_{K_0}\big] \leq C\xi(\ell_{K_0})S_{K_0},$$

and therefore

$$\mathbb{P}\Big(\sum_{v\in \mathrm{active}(\mathcal{E}_{K_0})} f(\mathcal{Z}[v,N]) < \mu S_{K_0}\Big|\mathcal{E}_{K_0}\Big) = O(\mathrm{e}^{-\frac{\log N}{\log\log N}}),$$

by (4.26) and the fact that H_{K_0} is a bounded multiple of S_{K_0}. Taking expectations and using the already established lower bound on the probability of a succesful exploration yields the desired bound of $\mathbb{P}(\sum_{v\in V(i)} f(\mathcal{Z}[v,N]) < \nu\xi(V(i))) \le \frac{\varepsilon}{24}$, for sufficiently large N. Summing up all error probabilities, we thus have shown that with probability execeeding $1-\varepsilon$,

$$d_{\lceil(1+\varepsilon)N\rceil}(U,V) \le 2D + 2 + \Big(\frac{1}{1+\alpha} + \eta\Big)\frac{\log N}{\log\log N} + 2k_0(\varepsilon)$$

which concludes the proof. □

5. Proof of Theorem 2

In this section we use a similar method as in the previous sections to describe the average distances in the Norros-Reittu model with i.i.d. random weights, and thus prove Theorem 2. The technical details are considerably easier in this case, and some parts of the proof which proceed in direct analogy to the preferential attachment case will only be sketched.

We first state some well known facts about heavy tailed i.i.d. weight sequences.

Proposition 5.1 (Asymptotics of weights)**.** *Let $(W_i)_{i\ge1}$ be an i.i.d. sequence satisfying*

$$\mathbb{P}(W_1 \ge k) = k^{-2}(\log k)^{2\alpha+o(1)},$$

and denote by F_n the distribution function of the n-th power W_1^n of the weights. For every $\varepsilon\in(0,1)$ there is a subset Ω_ε of the space of all infinite weight sequences with $\mathbb{P}(\Omega_\varepsilon) > 1-\varepsilon$ and positive constants C_1, C_2, C_3 and c_2 such that on Ω_ε the following conditions are satisfied

$$\max_{1\le i\le N} W_i \le C_1\big(\tfrac{1}{1-F_1}\big)^{-1}(N), \tag{5.1}$$

$$c_2 \le \frac{\sum_{i=1}^N W_i^2 - J(N)}{\big(\frac{1}{1-F_2}\big)^{-1}(N)} \le C_2, \tag{5.2}$$

$$\sum_{i=1}^N W_i^3 \le C_3\big(\tfrac{1}{1-F_3}\big)^{-1}(N), \tag{5.3}$$

where the generalised inverse of a monotone function is chosen to be left-continuous and

$$J(N) := N\mathbb{E}[W_1^2 \mathbb{1}\{W_1^2 \le \big(\tfrac{1}{1-F_2}\big)^{-1}(N)\}].$$

Proof. Inequality (5.1) is a direct consequence of the weak convergence of the rescaled maximum weight to the Fréchet distribution (see e.g. [Res87, Chapter I]). The relations (5.2) and (5.3) follow from weak convergence of rescaled partial sums to stable random variables with positive support (see e.g. [Res07, Corollary 7.1] for a stronger functional version). □

5.1. Proof of the lower bound.

It is now straightforward to deduce a first moment upper bound on the probability of existence of short paths in $\mathcal{H}_N$.

Proposition 5.2 (Lower bounds on distances in NR)**.** *Let $\mathcal{H}_N$ denote a Norros-Reittu network with weight distribution satisfying* (2.3)*, then for every $\delta\in(0,\frac{1}{1+2\alpha})$ and independently and uniformly chosen vertices $U,V\in\mathcal{H}_N$,*

$$d_N(U,V) \ge \Big(\frac{1}{1+2\alpha} - \delta\Big)\frac{\log N}{\log\log N} \quad \textit{with high probability as } N\to\infty.$$

Proof. We use Lemma 3.8 conditionally on the sequence $W_1, W_2, \dots$ of weights, and given N we relabel the vertices of $\mathcal{H}_N$ in decreasing order of weight and denote by $W^{(1)} \geq \cdots \geq W^{(N)}$ the order statistic of the first N weights. It is sufficient to verify the conditions of the lemma, for any $\varepsilon \in (0,1)$, on a subset Ω_ε of the space of all weight sequences with $\mathbb{P}(\Omega_\varepsilon) \geq 1-\varepsilon$. Conditional independence of edges immediately yields (3.13) with $\kappa_N = 1$. Let F be the distribution function of W_1 and, given a small $\delta > 0$ to be fixed later, pick $\Psi_N = (\log N)^{2\alpha + o(1)}$ such that, for sufficiently large N, and all $v \in \{1, \dots, N\}$,

$$p(v) := 1 - F\Big(\sqrt{\tfrac{N}{v}\Psi_N}\Big) \leq \delta \frac{v}{N}.$$

Denoting $L_N := \sum_{n=1}^N W_n \sim N\,\mathbb{E}W_1$, the conditional connection probabilities satisfy

$$\mathbb{P}(v \leftrightarrow w) \leq \frac{W^{(v)}W^{(w)}}{L_N}. \tag{5.4}$$

Hence, to show (3.13) it is sufficient to find $C(\varepsilon) > 0$ such that

$$W^{(v)} \leq C(\varepsilon)\sqrt{\frac{N}{v}\Psi_N}, \quad \text{for all } 1 \leq v \leq N, \tag{5.5}$$

with probability exceeding $1 - 2\varepsilon$. Let $S_N^{(v)}$ the number of weights $W_1, \dots, W_N$ exceeding $\sqrt{(N/v)\Psi_N}$. The random variable $S_N^{(v)}$ is binomial with parameters N and $p(v)$, hence Bernstein's inequality gives, for suitably fixed $\delta > 0$,

$$\mathbb{P}(S_N^{(v)} > 2v) \leq \exp\Big(-\frac{v^2}{2\mathrm{Var}S_N^{(v)} + \frac{2}{3}v}\Big) \leq \mathrm{e}^{-\frac{3}{8}v}.$$

Let M such that $\sum_{v=M}^\infty \mathrm{e}^{-\frac{3}{8}v} < \varepsilon$. Then (5.5) holds for all $v \geq 2M$ with probability exceeding $1-\varepsilon$. Increasing $C(\varepsilon)$ if necessary we get (5.5) with probability exceeding $1-2\varepsilon$. Application of Lemma 3.8 hence yields Proposition 5.2 as $\log \Psi_N = \big(2\alpha + o(1)\big)\log\log N$ and $\kappa_N = 1$. □

5.2. **Proof of upper bounds.** We now prove the upper bound in Theorem 2.

Proposition 5.3 (Upper bound on distances in NR)**.** *Let $\mathcal{H}_N$ be a Norros-Reittu network with weight distribution satisfying* (2.3)*. Consider vertices U, V chosen independently and uniformly at random from the largest component $\mathcal{C}_N \subset \mathcal{H}_N$. Then, for any $\delta \in (0, \frac{1}{1+2\alpha})$,*

$$d_N(U,V) \leq \Big(\frac{1}{1+2\alpha} + \delta\Big)\frac{\log N}{\log\log N} \quad \textit{with high probability as } N \to \infty.$$

The result can be obtained by a straightforward adaptation of the proof of [Hof16, Theorem 3.22], which uses the second moment method in combination with path counting techniques. For the closely related Chung-Lu model with deterministic weights, a related result is [CL06, Theorem 7.9], the proof of which also works in our setting. We provide a sketch of a proof relying on similar arguments as given in Section 4 for the preferential attachment network.

For $H \subset [N]$ we denote by $W(H) = \sum_{v \in H} W_v$ the *total weight* of H. As for the preferential attachment graph, the neighborhood of a uniformly chosen vertex $V \in \mathcal{H}_N$ converges in distribution to a random tree $\mathfrak{S}$. This tree can be obtained by a *mixed Poisson branching process*, see [NR06]. Denoting by $p(W)$ the survival probability of $\mathfrak{S}$, we get $\lim \frac{\#\mathcal{C}_N}{N} = p(W)$ in probability, see [Hof16, Section 3.1.].

The following facts are instrumental for our argument.

Lemma 5.4. *Choose $V \in [N]$ uniformly. For every $\varepsilon \in (0, p(W)), s_0 > 0$ there exists $k_0 > 0$, such that $\mathbb{P}(W(\{v \in [N] : d_N(V,v) = k_0\}) \geq s_0) \geq p(W) - \varepsilon$, for sufficiently large N.*

Proof. This follows from local weak convergence to $\mathfrak{S}$ and the fact that the offspring distribution of the branching process generating $\mathfrak{S}$ has infinite mean in every generation $k \geq 2$. □

Lemma 5.5. *Fix $M = \lceil \log N^R \rceil$ for some fixed $R > 0$ and let core$_N$ denote the subgraph of $\mathcal{H}_N$ induced by the M vertices with the largest weights. Then core$_N$ is bounded in $\mathcal{H}_N$ with high probability, as $N \to \infty$.*

Proof. Given N we relabel the vertices of $\mathcal{H}_N$ in decreasing order of weight and denote by $W^{(1)} \geq \cdots \geq W^{(N)}$ the order statistic of the first N weights. Fix $\varepsilon > 0$ and $\delta \in (0, \alpha)$. Then $L_N := \sum_{i=1}^N W_i \sim N\,\mathbb{E}W_1$, and

$$W^{(v)} \geq \sqrt{\frac{N}{M}} \big(\log \tfrac{N}{M} \big)^{\alpha - \delta}, \text{ for all } v \in [M],$$

on a subset Ω_ε with probability exceeding $1 - \varepsilon$, by a standard extreme value calculation, using e.g. [LLR83, Theorem 2.5.2]. Given the weights, each pair of vertices $v, w \in \text{core}_N$ independently is connected with probability at least

$$1 - \mathrm{e}^{-\frac{(W^{(M)})^2}{L_N}} \geq \frac{\big(\log \frac{N}{M} \big)^{2\alpha - 2\delta}}{3M\mathbb{E}W_1} =: p(M,N).$$

Now coupling to an Erdős-Rényi graph $\mathcal{G}(M, p(N,M))$ and [Bol01, Corollary 10.12] yield the boundedness of the diameter. □

Lemma 5.6. *If $V_1, V_2 \subset [N]$ are disjoint sets with total weights satisfying*

$$\lim_{N \to \infty} \frac{1}{N} W(V_1) W(V_2) = \infty \textit{ in probability,}$$

then they are connected with high probability in $\mathcal{H}_N$.

Proof. By conditional independence, $\mathbb{P}(V_1 \not\leftrightarrow V_2) = \mathrm{e}^{-\frac{W(V_1)W(V_2)}{L_N}}$, from which the result follows since $\frac{1}{L_N} W(V_1) W(V_2)$ diverges to infinity, in probability. □

Proof of Proposition 5.3. In view of Lemmas 5.4 to 5.6 it is sufficient to show that a truncated exploration in $\mathcal{H}_N$ started in a configuration $\mathcal{E}_0$ of large initial weight S_0 with high probability, as $N \to \infty$, reaches a configuration $\mathcal{E}_k$ satisfying

$$S_k = W(\text{active}(\mathcal{E}_k)) \geq \frac{\sqrt{N}}{(\log N)^R}$$

in less than K stages, where $R, \delta > 0$ are fixed and

$$K = \Big(\frac{1}{2 + 4\alpha} + \delta \Big) \frac{\log N}{\log \log N}.$$

We truncate the exploration in the following way: at stage k, we only investigate connections between active vertices and vertices of weight at most w_{k+1}, where $(w_k)_{k \geq 1}$ is a superexponentially growing sequence specified below. Since we would like to condition on the weights, we start by demonstrating that almost all weight sequences have certain properties. Let $(A_k)_{k=0}^K$ denote a partition of the set $[1, \sqrt{N}(\log N)^{2\alpha})$ into K nonoverlapping intervals $A_k = [a_k, a_{k+1})$ of equal length. Applying Lemma 4.2, and a brief calculation we may assume that $W_1, \dots, W_N$ satisfy,

$$\sum_{i=1}^{N} W_i^2 \mathbb{1}\{W_i \le w_k\} \ge \frac{1}{2}\mathbb{E}\sum_{i=1}^{N} W_i^2 \mathbb{1}\{W_i \le w_k\}, \quad \text{for } 1 \le k \le K, \tag{5.6}$$

as well as

$$\sum_{i=1}^{N} W_i^3 \mathbb{1}\{W_i \le w_k\} \le \frac{3}{2}\mathbb{E}\sum_{i=1}^{N} W_i^3 \mathbb{1}\{W_i \le w_k\}, \quad \text{for } 1 \le k \le K. \tag{5.7}$$

Fix $\varepsilon > 0$. Let $\mathcal{E}$ be a configuration obtained from an exploration of $\mathcal{H}_N$, $S = W(\text{active}(\mathcal{E}))$, $H = \text{active}(\mathcal{E}) \cup \text{dead}(\mathcal{E})$, $w > 0$ and $V = V(w) = \{v \in \text{veiled}(\mathcal{E}) : W_v \le w\}$. It is easy to see, using an appropriate coupling to a sum over independent weighted Bernoulli random variables and Lemma 4.2 that, as long as $wS = o(L_N)$,

$$W(\{v \in V : v \leftrightarrow \text{active}(\mathcal{E})\}) \ge \frac{\sum_{v\in V} W_v^2}{4L_N} S =: \nu(w, N)S, \tag{5.8}$$

conditional on $\mathcal{E}$ and the weight sequence, with probability at least

$$1 - \mathrm{e}^{-\frac{(\sum_{v\in V} W_v^2)^2}{4L_N \sum_{v\in V} W_v^3} S}. \tag{5.9}$$

Note that, by (5.2) and our choice of weight distribution,

$$\sum_{v\in V} W_v^2 \ge \sum_{v\in[N]} W_v^2 \mathbb{1}\{W_v \le w\} - \big(\max_{a\in H} W_a\big)W(H).$$

Hence choosing w_0 sufficiently large, setting $w_k = c(\delta, \varepsilon)\log N^{1+2\alpha-\eta(\delta)} w_{k-1},\ 1 \le k \le K$, for some appropriately chosen small values of $c(\delta, \varepsilon), \eta(\delta)$ and letting $V_k = V(w_k)$ in (5.8), we obtain that the weight S_k of the active vertices increases in each stage k of the exploration by a factor of at least $\nu(w_k, N) \ge c\log(w_k)^{2\alpha+1-\eta(\delta)}$, for some constant c which depends on δ and ε but not on N. A straightforward calculation now shows that the exploration satisfies

$$S_k = W(\text{active}(\mathcal{E}_k)) \ge \frac{\sqrt{N}}{(\log N)^R}$$

after at most K stages. Summing the error terms in (5.9) for the different stages using (5.6) and (5.7), we obtain for some constants c_1, c_2, which are independent of N,

$$\sum_{k=1}^{K} \mathrm{e}^{-\frac{(\sum_{v\in V_k} W_v^2)^2}{4L_N \sum_{v\in V_k} W_v^3} S_{k-1}} \le \sum_{k=1}^{K} \mathrm{e}^{-c_1 \frac{(\log w_k)^{4\alpha+2-2\eta(\delta)} N^2}{N^2 w_k (\log w_k)^{2\alpha+\eta(\delta)}} S_{k-1}} \le \sum_{k=1}^{K} \mathrm{e}^{-c_2(\log w_k)^{1-\eta(\delta)}} < \varepsilon,$$

as $N \to \infty$. This concludes the proof, since ε and δ where chosen arbitrarily. □

Appendix A. Further calculations for preferential attachment

The following lemma is used to prove Proposition 4.1. The proof relies on a coupling of local neighbourhoods in $\mathcal{G}_N$ with the 'idealised neighbourhood tree' $\mathfrak{T}$ introduced in [DM13, Section 1.3], in which vertices of the tree have positions on the negative real line. We denote by $\mathfrak{T}_k$ the k-th generation of $\mathfrak{T}$, and by $p(f)$ be the probability that $\mathfrak{T}$ is infinite.

Lemma A.1. *Let $\chi\colon [0, \infty) \to [1, \infty)$ be a an increasing function satisfying*

$$c \le \chi(x)\mathrm{e}^{-\frac{1}{2}x} \le C, \text{ for some } 0 < c \le C < \infty.$$

Denote by $\bar\chi\colon \mathfrak{T}\to[1,\infty)$ *the function defined on the vertices of* $\mathfrak{T}$ *by* $\bar\chi(v)=\chi(-x_v)$, *where* x_v *is the position of* $v\in\mathfrak{T}$ *on the negative real line. Then, for any* $s>0$, *almost surely conditional on* $\#\mathfrak{T}=\infty$ *there exists* $K\in\mathbb{N}$ *and* $A_K\subset\mathfrak{T}_K$ *such that*

$$\sum_{v\in A_K}\bar\chi(v)\geq s\max_{v\in A_K}\bar\chi(v).$$

Proof. On the event $\#\mathfrak{T}=\infty$ there exists, almost surely, a sequence (w_i) of vertices in $\mathfrak{T}$ with positions drifting to $-\infty$, see [DM13, Lemma 3.3]. We choose such a sequence adapted to the natural filtration of the branching process. For any $\eta>1$, the events that w_i has a child positioned in $[-2\eta,-\eta]$ are stochastically bounded from below by i.i.d. events of positive probability. Hence we find a vertex $v(1)$ of type ℓ in $\mathfrak{T}$ with position $x_{v(1)}\in[-2\eta,-\eta]$. Continuing inductively we construct an adapted sequence of vertices $v(i)$ of type ℓ in $\mathfrak{T}$ such that $x_{v(i)}\in[x_{v(i-1)}-2\eta,x_{v(i-1)}-\eta]$. Denote by $A(i)$ the set of offspring generated by $v(i)$ in $[x_{v(i)},0]$ and let $Y(i)=\sum_{v\in A(i)}\bar\chi(v)$. By definition of the underlying branching random walk, denoting by $(Z_t)_{t\geq 0}$ the idealised degree evolution process, we have

$$\mathbb{E}[Y_i\,|\,x_{v(i)}=x]=\int_0^{-x}\chi(-u-x)\mathbb{E}f(Z_u)\,\mathrm{d}u\geq c\mathrm{e}^{\frac12 x}\int_{-x_{v(i-1)}}^{-x}\mathrm{e}^{\frac12 u}\mathbb{E}f(Z_u)\,\mathrm{d}u.$$

Using the estimate $c'u^\alpha\mathrm{e}^{\frac12 u}\leq\mathbb{E}f(Z_u)\leq C'(u^\alpha\vee 1)\mathrm{e}^{\frac12 u}$, for all $u\geq 0$, which is a continuous analogue of Propositions 3.2 and 3.3 and may be shown in a similar fashion for our choice of attachment rule, we get a lower bound of

$$\mathbb{E}[Y_i\,|\,x_{v(i)}=x]\geq cc'\,\mathrm{e}^{\frac12 x}\int_{-x_{v(i-1)}}^{-x}\mathrm{e}^{\frac12 u}u^\alpha\mathrm{e}^{\frac12 u}\,\mathrm{d}u\geq c''\,(-x_{v(i-1)})^\alpha\mathrm{e}^{-\frac12 x},\tag{A.1}$$

for some constant $c''>0$ not depending on η. From (A.1) we get $i_0(s)\in\mathbb{N}$ such that

$$\mathbb{E}[Y_i\,|\,x_{v(i)}=x]\geq 2s\bar\chi(v(i)),\ \text{for all } i\geq i_0.\tag{A.2}$$

Calculating $\mathbb{E}[Y_i^2\,|\,x_{v(i)}=x]$ is slightly more subtle. We have

$$\mathbb{E}\Big[\sum_{v\in A(i)}\bar\chi^2(v)\Big|x_{v(i)}=x\Big]\leq C'\,(-x)^\alpha\mathrm{e}^{-\frac12 x},$$

for some constant $C'>0$, by a calculation similar to (A.1). Note that, by [DM13, Lemma 2.5], for any $u\geq 0$, we have $\mathbb{E}[f(Z_t)|\Delta Z_u=1]\leq\mathbb{E}[f(Z_t)|Z_0=1]\leq\frac{f(1)}{f(0)}\mathbb{E}f(Z_t)$, for all $t\geq u$. The offspring intensity of $v(i)$ on $[x_u,0]$ conditional on producing offspring in position x_u is thus bounded by a constant multiple of the unconditional intensity. This implies that

$$\mathbb{E}\Big[\sum_{\substack{u,v\in A(i)\\ u<v}}\bar\chi(u)\bar\chi(v)\,\Big|\,x_{v(i)}=x\Big]\leq C''\mathbb{E}\Big[\sum_{v\in A(i)}\bar\chi(v)\,\Big|\,x_{v(i)}=x\Big]^2\leq C'''\,(-x)^{2\alpha}\mathrm{e}^{-x},$$

by a similar calculation as above. Combining the previous two displays gives a bound on $\mathbb{E}[Y_i^2\,|\,x_{v(i)}=x]$. Using (A.2) and the Paley-Zygmund inequality, we infer

$$\mathbb{P}\big(Y_i\geq s\bar\chi(v(i))\,\big|\,x_{v(i)}=x\big)\geq\mathbb{P}\big(Y_i\geq\tfrac12\mathbb{E}[Y_i|x_{v(i)}=x]\big|x_{v(i)}=x\big)\geq\frac{\mathbb{E}[Y_i|x_{v(i)}=x]^2}{4\mathbb{E}[Y_i^2|x_{v(i)}=x]}.$$

The moment estimates and assumptions on $v(i)$ imply that, for some small constants $c,q>0$,

$$\mathbb{P}(Y_i\geq s\bar\chi(v(i))|x_{v(i)}=x)\geq c\Big(\frac{x_{v(i-1)}}{x}\Big)^{2\alpha}\geq q>0,$$

as soon as $i\geq i_0$. Clearly, $\max_{u\in A(i)}\bar\chi(i)$ is at most $\bar\chi(v(i))$, since $\bar\chi$ is decreasing. So each of the sets $A(i)$ has probability at least q of being a set with the desired property, and the assertion follows by conditional independence of the $A(i), i\geq i_0$. $\square$

Proof of Proposition 4.1. Denote the tree associated with the configuration $\mathcal{E}_k$ by T_k. The arguments of [DM13] imply that, with high probability, for any fixed k, the configuration T_k can be coupled to $\mathfrak{T}_k$ and the scores ξ defined on T_k can be associated to a function χ satisfying the conditions of Lemma A.1 such that $\xi = \bar{\chi}$ on corresponding vertices. The claim hence follows from Lemma A.1. $\square$

Lemma A.2. *There are $\eta \in (0,1)$ and $c > 0$ only depending on α such that for any choice of $A \subset \{\lceil 2\mathrm{e}^2 \rceil, \dots, N\}$ and $v_0 < \frac{\min A}{\mathrm{e}^2} \wedge \eta N$ satisfying*

$$\big(\log \tfrac{N}{\min A} \vee 1\big)^{\alpha} \xi^2(A) \le \frac{c}{2} N \big(\log \tfrac{N}{v_0}\big)^{\alpha+1}, \tag{A.3}$$

we have, for $V = \{v_0, \dots, N\} \setminus A$ and any $a \in A$,

$$\sum_{v \in V} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} \ge \frac{c}{2} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1}, \tag{A.4}$$

if N is suffciently large.

Proof. We set $\varepsilon_0 = \exp(-(2 + 2(\log(\mathrm{e}^{\alpha} + \frac{1}{2\mathrm{e}^2})))^{\frac{1}{1+\alpha}})$, $\eta = \varepsilon_0^{-2}$ and first assume that, for all $A \subset \{\lceil 2\mathrm{e}^2 \rceil, \dots, N\}$ and $v_0 < \frac{\min A}{\mathrm{e}^2} \wedge \eta N$,

$$\sum_{v=v_0}^{N} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} \ge \frac{1}{2^{\alpha+1}(\alpha+1)} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} \quad \text{for all } a \in A. \tag{A.5}$$

Then (A.3) implies that

$$\begin{aligned}
\sum_{v \in V} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} &\ge \sum_{v=v_0}^{N} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} - \sum_{v \in A} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} \\
&\ge \tfrac{1}{2^{\alpha+1}(\alpha+1)} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} - \big(\log \tfrac{N}{\min A} \vee 1\big)^{\alpha} \sum_{v \in A} \frac{1}{v} \\
&\ge \tfrac{1}{2^{\alpha+1}(\alpha+1)} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} - c_{(3.2)} \big(\log \tfrac{N}{\min A} \vee 1\big)^{\alpha} \tfrac{\xi^2(A)}{N} \\
&\ge \big(\tfrac{1}{2^{\alpha+1}(\alpha+1)} - c_{(3.2)} \tfrac{c}{2}\big) \big(\log \tfrac{N}{v_0}\big)^{\alpha+1} = \tfrac{c}{2} \big(\log \tfrac{N}{v_0}\big)^{\alpha+1},
\end{aligned}$$

setting $c := \frac{1}{2^{\alpha}(1+\alpha)(1+c_{(3.2)})}$. The conclusion of the lemma holds subject to (A.5).

Let $a \le \lfloor \varepsilon_0 N + 1 \rfloor$. Observe that

$$\sum_{v=v_0}^{N} \frac{1}{v} \big(\log \tfrac{a \vee v}{a \wedge v} \vee 1\big)^{\alpha} \ge \sum_{v=v_0}^{\lfloor \frac{a}{\mathrm{e}} \rfloor} \frac{1}{v} \big(\log \tfrac{a}{v}\big)^{\alpha} + \sum_{v=\lceil a\mathrm{e} \rceil}^{N} \frac{1}{v} \big(\log \tfrac{v}{a}\big)^{\alpha} =: \Sigma_1 + \Sigma_2.$$

As $x \mapsto \frac{1}{x} \big(\log \frac{a}{x}\big)^{\alpha}$ is decreasing, we find, using $v_0 < \frac{a}{\mathrm{e}^2}$ in the last step, that

$$\begin{aligned}
\Sigma_1 &\ge \int_{v_0}^{\lfloor \frac{a}{\mathrm{e}} \rfloor + 1} \frac{1}{x} \big(\log \tfrac{a}{x}\big)^{\alpha} \, \mathrm{d}x = \tfrac{1}{1+\alpha} \Big(\big(\log \tfrac{a}{v_0}\big)^{\alpha+1} - \big(\log \tfrac{a}{\lfloor \frac{a}{\mathrm{e}} \rfloor + 1}\big)^{\alpha+1} \Big) \\
&\ge \tfrac{1}{1+\alpha} \Big(\big(\log \tfrac{a}{v_0}\big)^{\alpha+1} - 1 \Big) \ge \tfrac{1}{2(1+\alpha)} \big(\log \tfrac{a}{v_0}\big)^{\alpha+1}.
\end{aligned}$$

The map $x \mapsto \frac{1}{x} \big(\log \frac{x}{a}\big)^{\alpha}$ has a unique maximum at $x = \mathrm{e}^{\alpha} a$, thus

$$\begin{aligned}
\sum_{v=\lfloor \mathrm{e}^{\alpha} \rfloor + 1}^{N} \tfrac{1}{v} \big(\log \tfrac{v}{a}\big)^{\alpha} &\ge \int_{\lfloor \mathrm{e}^{\alpha} \rfloor + 1}^{N+1} \tfrac{1}{x} \big(\log \tfrac{x}{a}\big)^{\alpha} \, \mathrm{d}x \ge \tfrac{1}{\alpha+1} \big(\big(\log \tfrac{N+1}{a}\big)^{\alpha+1} - \big(\log \tfrac{\lfloor \mathrm{e}^{\alpha} \rfloor + 1}{a}\big)^{\alpha+1} \big) \\
&\ge \tfrac{1}{\alpha+1} \big(\big(\log \tfrac{N+1}{a}\big)^{\alpha+1} - \big(\log(\mathrm{e}^{\alpha} + \tfrac{1}{a})\big)^{\alpha+1} \big)
\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}\sum_{v=\lceil a\mathrm{e}\rceil}^{\lfloor \mathrm{e}^\alpha a\rfloor} \frac{1}{v}\big(\log \tfrac{v}{a}\big)^\alpha &\geq 1\!\!1\{\lceil a\mathrm{e}\rceil \leq \lfloor \mathrm{e}^\alpha a\rfloor\}\int_{\lceil a\mathrm{e}\rceil-1}^{\lfloor \mathrm{e}^\alpha a\rfloor} \frac{1}{x}\big(\log \tfrac{x}{a}\big)^\alpha \,\mathrm{d}x \\ &\geq \tfrac{1\!\!1\{\lceil a\mathrm{e}\rceil \leq \lfloor \mathrm{e}^\alpha a\rfloor\}}{\alpha+1}\big(\big(\log(\mathrm{e}^\alpha - \tfrac{1}{a})\big)^{\alpha+1}-1\big)\end{aligned} \tag{A.7}$$

Combining (A.6) and (A.7), we get

$$\begin{aligned}\Sigma_2 &\geq \tfrac{1}{\alpha+1}\big(\big(\log \tfrac{N+1}{a}\big)^{\alpha+1} - \big(\log(\mathrm{e}^\alpha+\tfrac{1}{a})\big)^{\alpha+1} + 1\!\!1\{\lceil a\mathrm{e}\rceil \leq \lfloor \mathrm{e}^\alpha a\rfloor\}\big(\big(\log(\mathrm{e}^\alpha-\tfrac{1}{a})\big)^{\alpha+1}-1\big)\big) \\ &\geq \tfrac{1}{\alpha+1}\big(\big(\log \tfrac{N+1}{a}\big)^{\alpha+1} - \big(\log(\mathrm{e}^\alpha+\tfrac{1}{2\mathrm{e}^2})\big)^{\alpha+1} - 1\big) \geq \tfrac{1}{2(\alpha+1)}\big(\log\tfrac{N+1}{a}\big)^{\alpha+1},\end{aligned}$$

where we used the condition $a \leq (N+1)\exp(-(2+2(\log(\mathrm{e}^\alpha+\frac{1}{2\mathrm{e}^2})))^{\frac{1}{1+\alpha}})$ in the last step. Combining the estimates for Σ_1 and Σ_2 yields

$$\sum_{v=v_0}^{N}\frac{1}{v}\big(\log\tfrac{a\vee v}{a\wedge v}\vee 1\big)^\alpha \geq \tfrac{1}{2(\alpha+1)}\big(\big(\log\tfrac{a}{v_0}\big)^{\alpha+1} + \big(\log\tfrac{N+1}{a}\big)^{\alpha+1}\big) \geq \tfrac{1}{2^{\alpha+1}(\alpha+1)}\big(\log\tfrac{N}{v_0}\big)^{\alpha+1}$$

by convexity of $x\mapsto x^{\alpha+1}$. Now consider $a \geq \lceil \varepsilon_0 N\rceil$. We have

$$\begin{aligned}\sum_{v=v_0}^{N}\frac{1}{v}\big(1\vee\log\tfrac{a\vee v}{a\wedge v}\big)^\alpha &\geq \int_{v_0}^{\lceil\varepsilon_0 N\rceil}\frac{1}{x}\big(1\vee\log\tfrac{a}{x}\big)^\alpha\,\mathrm{d}x \geq \int_{v_0}^{\varepsilon_0 N}\frac{1}{x}\big(\log\tfrac{\varepsilon_0 N}{x}\big)^\alpha\,\mathrm{d}x \\ &= \tfrac{1}{\alpha+1}\big(\log\tfrac{\varepsilon_0 N}{x}\big)^{\alpha+1}.\end{aligned}$$

Since $(\log\frac{\varepsilon_0 N}{x})^{\alpha+1} \geq \frac{1}{K}(\log\frac{N}{v_0})^{\alpha+1}$ if and only if $v_0 \leq N\varepsilon_0^{(1-(\frac{1}{K})^{\frac{1}{1+\alpha}})^{-1}}$, we choose $K = 2^{\alpha+1}$ and the desired bound (A.5) follows. □

Lemma A.3. *For any $\alpha\geq 0, \delta\in(0,2\alpha+2)$ let*

$$k_0(\delta,\alpha) = \min\{k\geq 3 : \delta\log k \geq (2\alpha+2-\delta)k\log(1+\tfrac{1}{k})+1\} \tag{A.8}$$

then

$$\ell_k \leq N\mathrm{e}^{-(2\alpha+2-\delta)(k-k_0)\log k} \quad \textit{for all } k_0\leq k < k^*(N).$$

Furthermore, there is a constant $c>0$ depending only on s_0 such that

$$\ell_k \geq CN\mathrm{e}^{-(4\alpha+5)k(1\vee\log k)}, \quad \textit{for all } k.$$

Proof. We first show the upper bound by induction in k. For $k=k_0$ the assertion is trivially true as soon as N is large enough. Now assume that $\ell_k \leq N\mathrm{e}^{-(2\alpha+2-\delta)(k-k_0)\log k}$ for some $k<k^*(N)-1$ then we have, by definition of $(\ell_k)_{k\geq 1}$, that $\log\ell_{k+1} \leq \log\ell_k - (2\alpha+2)\log(\log N - \log\ell_k)+1$ and applying the induction hypothesis yields

$$\begin{aligned}\log\tfrac{\ell_{k+1}}{N} \leq -(2\alpha+2-\delta)(k-k_0)\log(k+1) + \big((2\alpha+2-\delta)(k-k_0)\log\tfrac{k+1}{k}+1\big) \\ -(2\alpha+2)\log\big((k+1)\tfrac{k}{k+1}(2\alpha+2)\log k\big).\end{aligned}$$

By (A.8) we have $\frac{k}{k+1}(2\alpha+2)\log k\geq 1$, hence

$$\begin{aligned}\log\tfrac{\ell_{k+1}}{N} \leq &-(2\alpha+2-\delta)(k+1-k_0)\log(k+1) \\ &+\big((2\alpha+2-\delta)(k-k_0)\log\tfrac{k+1}{k}+1-\delta\log(k+1)\big).\end{aligned}$$

The second term of the sum is negative by (A.8) and the induction is complete. The lower bound follows by a similar argument. □

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