Directed random walk on the directed percolation cluster

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Based on joint work in progress with Jiří Černý, Andrej Depperschmidt and Nina Gantert

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**General aim:**
Study/understand the space-time embedding of ancestral lineages in spatial models for populations with local density regulation (in particular, with non-constant local population sizes).
Outline

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Study/understand the space-time embedding of ancestral lineages in spatial models for populations with local density regulation (in particular, with non-constant local population sizes).

1. Directed percolation

2. Random walk on the cluster
   - A renewal structure

3. Locally regulated populations (and ancestral lineages)
Directed (site) percolation

\[ p \in (0, 1), \ \omega(x, n), \ x \in \mathbb{Z}^d, \ n \in \mathbb{Z} \ \text{i.i.d. Bernoulli}(p). \]
Interpretation: \( \omega(x, n) = 1 \): site \((x, n)\) is open, otherwise closed
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Let $U \subset \mathbb{Z}^d$ be a finite, $\mathbb{Z}^d$-symmetric set with $0 \in U$

$m < n$, $x, y \in \mathbb{Z}^d$: $(x, m) \rightarrow (y, n)$ if there exist $x = x_0, x_1, \ldots, x_{n-m} = y$
such that $x_i - x_{i-1} \in U$ and $\omega(x_i, m + i) = 1$ for $i = 1, \ldots, n - m$
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$C_0 := \{(y, n) : y \in \mathbb{Z}^d, n \geq 0, (0, 0) \rightarrow (y, n)\}$ is the (directed) cluster of the origin
There exists $p_c \in (0, 1)$ such that

$$\mathbb{P}(|C_0| = \infty) > 0 \quad \text{iff} \quad p > p_c.$$
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\[ \mathbb{P}(\{|C_0| = \infty\} > 0 \iff p > p_c. \]

If $p > p_c$, \[ \mathbb{P}(C_0 \text{ reaches height } n \mid |C_0| < \infty) \leq C e^{-cn} \text{ for some } c, C \in (0, \infty). \]
The discrete time contact process and directed percolation

$\eta_n(x)$, $n \in \mathbb{Z}_+$, $x \in \mathbb{Z}^d$ with values in $\{0, 1\}$. Site $x$ is generation $n$ is “inhabited” (or: “infected”) if $\eta_n(x) = 1$. 
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Dynamics: \( U \subset \mathbb{Z}^d \) finite, symmetric, \( p \in (0, 1) \).

Given \( \eta_n \), independently for \( x \in \mathbb{Z}^d \),

\[
\eta_{n+1}(x) = \begin{cases} 
1 & \text{w. prob. } p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U) \\
0 & \text{w. prob. } 1 - p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U) 
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The discrete time contact process

Self duality: For $A, B \subset \mathbb{Z}^d$

$$\mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in B \mid \eta_0(\cdot) = 1_A(\cdot))$$

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\]

Stationary process:
For \( p > p_c \), there is a (unique extremal) non-trival stationary distribution. Informally, \( \eta_0^{\text{stat}}(x) = 1 \) iff \( \mathbb{Z}^d \times \{-\infty\} \rightarrow (x, 0) \)
The discrete time contact process ...

... as a locally regulated population model

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Possible interpretation for ancestry:
In generation \( n + 1 \), each site \( x \) is inhabitable with probability \( p \).
If \( \eta_n(y) = 1 \) of some \( y \in x + U \), the particle at \( y \) in gen. \( n \) puts an offspring at \( x \).
If several \( y \) are eligible, one is chosen at random.

Thus, individuals in "sparsely populated" regions have a higher reproduction probability.
This is particularly evident in multitype version, where occupied sites carry a type, e.g. from \((0, 1)\), and offspring inherit parents' type.
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An ancestral line in the discrete time contact process

\[ p > p_c, \ (\eta_n^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z}) \text{ stationary DCP, assume } \eta_0^{\text{stat}}(0) = 1. \]

Let \( X_n = \text{position of the ancestor of the individual at the (space-time) origin } n \text{ generations ago.} \)
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$$\{y \in \mathbb{Z}^d : y - x \in U, \eta_{-n-1}(y) = 1\} \ (\neq \emptyset).$$
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To avoid lots of $-$ signs later, put $\xi_n(x) := \eta_{-n}^{\text{stat}}(x)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$.

Note: $\xi_n(x) = 1$ iff “$(x, n) \to \mathbb{Z}^d \times \{+\infty\}”$
Directed random walk on the supercritical directed cluster

\[ \omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}, \text{i.i.d. Bernoulli}(p), p > p_c \]

\[ \xi_n(x) = 1 \text{ iff } (x, n) \to (y, k) \text{ for infinitely many } (y, k) \quad \left( \text{“(}x, n\text{) \to +\infty”} \right) \]

Write \( C := \{(y, m) : \xi_m(y) = 1\} \), \( U(x, n) := (x + U) \times \{n + 1\} \)
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Let \( X_0 = 0 (\in \mathbb{Z}^d) \),

\[ \mathbb{P}(X_{n+1} = y \mid \xi, X_n = x, X_{n-1} = x_{n-1}, \ldots X_1 = x_1) = \frac{1(y \in U(x, n) \cap C)}{|U(x, n) \cap C|} \]

(with some arbitrary setting if \( U(x, n) \cap C = \emptyset \), we will later consider \( \xi \) under \( \mathbb{P}(\cdot \mid (0, 0) \in C) \))
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**Aim**: Understand the long-time behaviour of \((X_n)\). Is it similar to “ordinary” random walk?

**Note**: For the voter model (\( \approx \) contact process when no empty sites are allowed), ancestral lines are literally (coalescing) random walks.
Remark.

$\{X_n\}$ is a random walk in space-time random environment (given by $\xi$).

Random walks in random environments and recently also random walk in space-time random environments have received considerable attention (see e.g. Firas Rassoul-Agha’s homepage http://www.math.utah.edu/~firas/Research/)

As far as we know, none of the general techniques developed so far in this context is applicable:

- $\{X_n\}$ is not uniformly elliptic.
- $\xi$ is complicated: not i.i.d., nor is $\{\xi_n(x)\}_{n=0,1,...}$ for fixed $x$ a Markov chain.
- The abstract conditions from Dolgopyat, Keller and Liverani (2008) appear very hard to verify.
“Ancestor” ordering

For $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$ let $	ilde{\omega}(x, n) = (\tilde{\omega}(x, n)[1], \tilde{\omega}(x, n)[2], \ldots, \tilde{\omega}(x, n)[|U|])$ an independent uniform permutation of $U(x, n) = (x + U) \times \{n + 1\}$. 
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$\Gamma^k_{(x, n)} :=$ set of all $k$-step (directed) paths

$$\gamma = ((x_0, n), (x_1, n + 1), \ldots, (x_k, n + k))$$

starting at $x_0 = x$ whose steps begin at open sites, i.e., $\omega(x_i, n + i) = 1$ for $i = 0, 1, \ldots, k - 1$. 

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**Order** $\Gamma^k_{(x, n)} \ni \gamma, \gamma' = ((x = x'_0, n), (x'_1, n + 1), \ldots, (x'_k, n + k))$:

$1 \leq \ell (< k)$ the minimal value s.th. $x_\ell \neq x'_\ell$, then

$\gamma \prec \gamma'$ if $x_\ell$ has a smaller index than $x'_\ell$ in $\tilde{\omega}(x_{\ell-1}, n + \ell - 1)$.

$A^{(1)}_{(x, n); k} :=$ (spatial) endpoint of the smallest path in $\Gamma^k_{(x, n)}$ (if $\Gamma^k_{(x, n)} \neq \emptyset$) 

(first (potential) ancestor $k$ generations ago of site $(x, n)$)
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an independent uniform permutation of \( U(x, n) = (x + U) \times \{n + 1\} \).

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1 \( \leq \ell (\prec k) \) the minimal value s.th. \( x_\ell \neq x'_\ell \), then
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\( A^{(1)}_{(x, n); k} := \) (spatial) endpoint of the smallest path in \( \Gamma^k_{(x, n)} \) (if \( \Gamma^k_{(x, n)} \neq \emptyset \))
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Remarks. 1) Construction of \( A^{(1)}_{(x, n); k} \) measurable w.r.t.
\[
\sigma(\omega(y, i), \tilde{\omega}(y, i) : y \in \mathbb{Z}^d, n \leq i < n + k)
\]
2) Discrete time analogue of Neuhauser (1992)
Ancestor ordering and regeneration

\[ \kappa(x, n) := \tilde{\omega}(x, n) \left[ \min\{i : \xi_{n+1}(\tilde{\omega}(x, n)[i]) = 1\} \wedge |U| \right] \] (with min \( \emptyset := +\infty \))

\( \kappa(x, n) \) is uniformly distributed on \( U(x, n) \cap C \) if the latter is not empty and uniformly distributed on \( U(x, n) \) otherwise.
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On \( A_0 := \{(0, 0) \in C\} \)

\[ X_0 = 0, \quad X_{n+1} := \kappa(X_n, n), \quad n = 1, 2, \ldots \]

is (a version of) the directed random walk on \( C \), and \( X_k = A_{(0,0);k}^{(1)} \) if \( \xi_k(A_{(0,0);k}^{(1)}) = 1 \).
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**Regeneration times:**

\( T_0 := 0, \quad Y_0 := 0, \)

\( T_1 := \min\{n > 0 : \xi_n(A^{(1)}_{(0,0);n}) = 1\}, \quad Y_1 := A^{(1)}_{(0,0);T_1} = X_{T_1} \),

then \( T_2 := T_1 + \min\{n > 0 : \xi_{T_1+n}(A^{(1)}_{(Y_1, T_1);n}) = 1\} \),

\( Y_2 := A^{(1)}_{(Y_1, T_1);T_2-T_1} = X_{T_2} \), etc.
Proposition

\[ ((Y_i - Y_{i-1}, T_i - T_{i-1}))_{i \geq 1} \text{ is i.i.d. under } \mathbb{P}(\cdot \mid A_0), \text{ } Y_1 \text{ is symmetrically distributed. There exist } C, c \in (0, \infty), \text{ such that} \]

\[ \mathbb{P}(||Y_1|| > n \mid A_0), \mathbb{P}(\tau_1 > n \mid A_0) \leq Ce^{-cn} \text{ for } n \in \mathbb{N}. \]
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Remark

Regeneration structure and proof analogous to Kuczek (1989) and adaptation by Neuhauser (1992):

For tails of \( T_1 - T_0 \) use “restart” argument (to remove conditioning on \( A_0 \)) and the fact that finite clusters are small, i.i.d. property follows from the fact that the ancestor ordering construction uses disjoint time-slices.
LLN and annealed CLT for directed walk on the cluster

**Corollary**

\[ \mathbb{P} \left( \frac{1}{n} X_n \rightarrow 0 \ \bigg| \ A_0 \right) = 1 \quad \text{and} \quad \mathbb{P} \left( \frac{1}{n} X_n \rightarrow 0 \ \bigg| \ \omega \right) = 1 \quad \text{for} \ \mathbb{P}(\cdot \mid A_0)\text{-a.a.} \ \omega, \]

there exists \( \sigma \in (0, \infty) \) s.th.

\[ \lim_{n \to \infty} \mathbb{E} \left[ f \left( \frac{1}{\sigma \sqrt{n}} X_n \right) \ \bigg| \ A_0 \right] = \mathbb{E} \left[ f(Z) \right] \]

for any continuous bounded \( f : \mathbb{R}^d \to \mathbb{R} \), where \( Z \) is \( d \)-dimensional standard normal.
A quenched CLT

**Theorem**

Let $d \geq 3$, $p > p_c$.

$$\lim_{n \to \infty} \mathbb{E}\left[f\left(\frac{1}{\sigma \sqrt{n}} X_n\right) \mid \omega\right] = \mathbb{E}[f(Z)]$$

for $\mathbb{P}(\cdot \mid A_0)$-a.a. $\omega$

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(An invariance principle holds as well.)
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**Conjecture**

An analogous statement holds for $d = 1, 2$. 
Two walks on the same cluster

\((X_n), (X'_n)\) two independent directed walks on the same supercritical directed cluster \(\xi\) (i.e. using the same \(\omega\)'s, but independent \(\tilde{\omega}\)'s resp. \(\tilde{\omega}'\).)

**Proposition**

Let \(d \geq 3\). There exists \(b > 0\) such that for \(f, g \in C_b(\mathbb{R}^d)\)

\[
\left| \mathbb{E} \left[ f \left( \frac{1}{\sigma \sqrt{n}} X_n \right) g \left( \frac{1}{\sigma \sqrt{n}} X'_n \right) \mid \mathcal{A}_0 \right] - \mathbb{E} [f(Z)] \mathbb{E} [g(Z)] \right| \leq \frac{C_{f,g}}{n^b},
\]

in particular \(\mathbb{E} \left[ f \left( \frac{1}{\sigma \sqrt{n}} X_n \right) \mid \omega \right] \rightarrow \mathbb{E} [f(Z)]\) in \(L^2(\mathbb{P}(\cdot \mid \mathcal{A}_0))\).
Two walks on the same cluster

\((X_n), (X'_n)\) two independent directed walks on the same supercritical directed cluster \(\xi\) (i.e. using the same \(\omega\)'s, but independent \(\tilde{\omega}\)'s resp. \(\tilde{\omega}'\).)

**Proposition**

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\[
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in particular \(\mathbb{E} \left[ f \left( \frac{1}{\sigma \sqrt{n}} X_n \right) \right] \xrightarrow{\omega} \mathbb{E} \left[ f(Z) \right]\) in \(L^2(\mathbb{P}(\cdot \mid A_0))\).

Proof uses joint regeneration structure of two walks on the same cluster:
Exponential mixing of \(\xi\) and the fact that two independent walks in \(d \geq 3\) eventually separate allows to couple with two walks on independent copies \(\xi\) and \(\xi'\) with high probability.

From Prop., obtain first quenched CLT for \((X_n)\) along subsequence, then additional concentration argument.
Two walks on the same cluster

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**Remarks**

1) Variation where \((X_n)\) and \((X'_n)\) coalesce upon meeting is of interest in mathematical population genetics.

Ex.: In multi-type contact process in equilibrium in \(d \geq 3\),

\(\mathbb{P}\)\(\text{(two ind. sampled at distance } x \text{ have same type)} \sim C \cdot x^{2-d}\).

2) (Some) analogous arguments for the continuous-time case by Neuhauser (1992) and Valesin (2010).
A spatial logistic model

Particles “live” in $\mathbb{Z}^d$ in discrete generations, 
$\eta_n(x) = \# \text{ particles at } x \in \mathbb{Z}^d \text{ in generation } n.$

Given $\eta_n$, 
each particle at $x$ has $\text{Poisson}(m - \sum_z \lambda_{z-x} \eta_n(z))$ offspring, 
$m > 1, \lambda_z \geq 0, \lambda_0 > 0$, finite range.
Children take an independent random walk step to $y$ with probability $p_{y-x}$, 
$p_{xy} = p_{y-x}$ symmetric, aperiodic finite range random walk kernel on $\mathbb{Z}^d$. 
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$p_{xy} = p_{y-x} \text{ symmetric, aperiodic finite range random walk kernel on } \mathbb{Z}^d.$

Given $\eta_n$,

$\eta_{n+1}(y) \sim \text{Poi} \left( \sum_x p_{y-x} \eta_n(x) \left( m - \sum_{z} \lambda_{z-x} \eta_n(z) \right)_+ \right), \text{ independent}$
Survival and complete convergence

Theorem (B. & Depperschmidt, 2007)
Assume $m \in (1, 3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

$(\eta_n)$ survives for all time globally and locally with positive probability for any non-trivial initial condition $\eta_0$.

Given survival, $\eta_n$ converges in distribution to its unique non-trivial equilibrium.

Starting from any two initial conditions $\eta_0, \eta'_0$, copies $(\eta_n), (\eta'_n)$ can be coupled such that if both survive, $\eta_n(x) = \eta'_n(x)$ in a space-time cone.
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Proof uses that corresponding deterministic system

$$\zeta_{n+1}(y) = \sum_x p_{y-x}\zeta_n(x) \left( m - \sum_z \lambda_{z-x} \zeta_n(z) \right)_+$$

has unique non-triv. fixed point

plus coarse-graining, lots of comparisons with directed percolation.
Locally regulated populations (and ancestral lineages)

**Coupling**

$m = 1.5$, $p = (1/3, 1/3, 1/3)$, $\lambda = (0.01, 0.02, 0.01)$
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Locally regulated populations (and ancestral lineages)

Coupling

\[
m = 1.5, \ p = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \ \lambda = \left( 0.01, 0.02, 0.01 \right)
\]
Ancestral lines

Given stationary \( (\eta_n^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d) \), cond. on \( \eta_0^{\text{stat}}(0) > 0 \), sample an individual from space-time origin \((0, 0)\) (uniformly)

Let \((X_n)\) position of her ancestor \(n\) generations ago:

Given \(\eta^{\text{stat}}\) and \(X_n = x\), \(X_{n+1} = y\) w. prob.

\[
p_{x-y} \eta_{n-1}^{\text{stat}}(y) \left( m - \sum_z \lambda_{z-y} \eta_{n-1}^{\text{stat}}(z) \right) + \sum_{y'} p_{x-y'} \eta_{n-1}^{\text{stat}}(y') \left( m - \sum_z \lambda_{z-y'} \eta_{n-1}^{\text{stat}}(z) \right)
\]
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\[
p_{x-y} \eta_{n-1}^{\text{stat}}(y) \left( m - \sum_z \lambda_{z-y} \eta_{n-1}^{\text{stat}}(z) \right)_+ \quad \sum_{y'} p_{x-y'} \eta_{n-1}^{\text{stat}}(y') \left( m - \sum_z \lambda_{z-y'} \eta_{n-1}^{\text{stat}}(z) \right)_+
\]

Hopeful theorem in progress ...

If \(m \in (1, 3), 0 < \lambda_0 \ll 1, \lambda_z \ll \lambda_0\) for \(z \neq 0\), there is a regeneration construction for \((X_n)\).

This again yields LLN and CLT for the ancestral line of an individual drawn from equilibrium.
Thank you for your attention!