Approximate Lifshitz law for the zero-temperature stochastic Ising model in dimension $d \geq 4$

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Game rules

Given $L \in \mathbb{N}$, set $\Lambda_L = \{1, \ldots, L\}^d \subset \mathbb{Z}^d$ we consider a Markov chain on $\Omega_L = \{+1, -1\}^{\Lambda_L}$ ($\sigma = (\sigma_x)_{x \in \Lambda_L}$ denote a configuration in $\Omega_L$) $(\sigma(n))_{n \geq 0}$ with the following rule:

(i) One starts from $\sigma(0) \equiv -1$ ($\sigma_x(0) = -1$) for all $x \in \Lambda_L$.

(ii) To get $\sigma(n+1)$ from $\sigma(n)$:

(a) One chooses $X_n$ uniformly at random in $\Lambda_L$.

(b) On sets $\sigma_y(n+1) = \sigma_y(n)$ for all $y \neq X_n$.

(c) $\sigma_{X_n}(n+1)$ is equal to $+1$ if the majority of neighbors of $X_n$ satisfies $\sigma(n)_y = +1$, $-1$ if a majority of neighbor have spin $-1$. $\pm 1$ with proba $1/2$ each, if half of neighbors are $+$ and the other half $-$.

If a neighbor of $X_n$ is not in $\Lambda_L$ is not in $\Lambda_L$ its spin is considered to be $+$ at all time: $\sigma_y(n) = +1$, $\forall n \forall y \in \partial \Lambda_L = \{z \in \mathbb{Z}^d \setminus \Lambda_L \mid \exists x, y \setminus x\}$. 
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What it looks like
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What it looks like

![Diagram of a 4D Ising model](image-url)
What it looks like

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[Diagram of a 4D Ising model]
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What it looks like
Continuous time version

For commodity, we consider a continuous time version of this process $(\sigma(t))_{t \geq 0}$ where the update are done according to a Poisson Point Process of rate $L^d = |\Lambda_L|$ (each site is updated with rate one independently)

$$\forall n, \forall t \in [\tau_n, \tau_{n+1}), \sigma(t) = \sigma(n),$$

where $\tau_{n+1} - \tau_n$ are IID exponentials with mean $L^{-d}$. 
Typical behavior for large $L$
Typically behavior for large $L$
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Typical behavior for large $L$
Typicall behavior for large $L$
Typical behavior for large $L$
Typical behavior for large $L$
Typicall behavior for large $L$
Typicall behavior for large $L$
Typical behavior for large $L$
Questions

- How much time do you need the set of $\ldots$ to disappear (for large $L$).
- What is the behavior of the $\ldots$ droplet.
Non-zero temperature model

Stochastic Ising Model at temperature $\beta^{-1} > 0$: Given $L \in \mathbb{N}$, Set $\Lambda_L = \{1, \ldots, L\}^d \subset \mathbb{Z}^d$ we consider a Markov chain on $\Omega_L = \{+1, -1\}^{\Lambda_L}$ ($\sigma = (\sigma_x)_{x \in \Lambda_L}$ denote a configuration in $\Omega_L$) $(\sigma(n))_{n \geq 0}$ with the following rule:

(i) One starts from $\sigma(0) \equiv -1$ ($\sigma_x(0) = -1$) for all $x \in \Lambda_L$.
(ii) To get $\sigma(n+1)$ from $\sigma(n)$:
   (a) One chooses $X_n$ uniformly at random in $\Lambda_L$.
   (b) Set $\sigma^{X_n, +}$, (resp. $\sigma^{X_n, -}$) be the configuration defined by $\sigma^{X_n, +}_y = \sigma_y(n)$ and $\sigma^{X_n, +}_x = +1$, (resp. $-1$). One sets

$$
\sigma(n+1) = \sigma^{X_n, +}_X \quad \text{with proba} \quad \frac{e^{-\beta H(\sigma^{X_n, +})}}{e^{-\beta H(\sigma^{X_n, +})} + e^{-\beta H(\sigma^{X_n, -})}},
$$

$$
\sigma(n+1) = \sigma^{X_n, -}_X \quad \text{with proba} \quad \frac{e^{-\beta H(\sigma^{X_n, -})}}{e^{-\beta H(\sigma^{X_n, +})} + e^{-\beta H(\sigma^{X_n, -})}}.
$$

where

$$
H(\sigma) := -\sum_{x \in \Lambda_L} \sum_{y \sim x} \sigma_x \sigma_y.
$$
Question for the non-zero temperature model

The only stationary measure for the dynamic is

\[ \mu^\beta_L(\sigma) := \frac{1}{Z^\beta_L} e^{-\beta H(\sigma)} \]

**Question**

- Time needed to reach equilibrium. Call \( \nu_t \) the law of \( \sigma(t) \).

\[ T_{\text{mix}}(\varepsilon) := \inf\{ t : \| \nu_t - \mu^\beta_L(\sigma) \|_{TV} \leq \varepsilon \} \]

- How does it reaches it (harder to define such a thing as “the minus droplet” in that case)
Why study dynamics

- They can be a computationnaly cheap way to sample Gibbs-measure (Knowledge of mixing time is then crucial for sampling).
- In the case the case of Ising model, they model physical evolution of ferromagnet.
1. Introduction

2. Review of results and predictions
   - The high temperature case
   - Lifshitz law at low temperature
   - Zero-temperature for $d = 2, 3$

3. Result and proof
High temperature

At high temperature \((\beta < \beta_c)\), there is no long range correlation between spins in the equilibrium measure. Morally different zones of finite volume should come to equilibrium independently...

Theorem (Lubetzky, Sly ’10)

For \(\beta \) small enough (when strong spatial mixing holds), for any \(\varepsilon\), for the dynamic with periodic boundary condition,

\[
T_{\text{mix}}^L(\varepsilon) = c(\beta) \log L(1 + o(1)).
\]
Critical temperature case

Critical temperature case is a priori very difficult as one does not know with precision the behavior of the "static" Ising model. Two a recent breakthrough on in two-dimension (bounds on crossing probabilities for FK-percolation by Duminil-Copin, Hongler, Nolin '10) allowed to prove the following,

**Theorem (Lubetzky, Sly '10)**

For \( d = 2 \), for the dynamic with arbitrary boundary condition,

\[
L^{C_1} \leq T_{\text{mix}}^L(\varepsilon) \leq L^{C_2}.
\]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

$T = 0$

All spin $-$
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = C \log L \]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = cL^2/10 \]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = \frac{3cL^2}{10} \]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = cL^2 / 2 \]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = \frac{4cL^2}{5} \]
Low Temperature case: Phenomenology (The Lifshitz Law [Lifshitz’76])

\[ T = cL^2 \]
Mean curvature motion
Problem with heuristics

- For finite $\beta$ it is not trivial even to define what the interface is.
- The network is anisotropic and this should play a role in the way to read \textit{mean curvature}.
- In general mathematicians are very far to be able to prove such results (attempts to make this rigorous in dimension for $d = 2$, $\beta = \infty$ [Spohn '93], [Cerf Louhichi '07]). The local drift of the interface (after passing to the limit) should be

\[
\frac{1}{2(|\sin(\theta)| + |\cos(\theta)|)^2} \times \text{curvature.}
\]  

(3)

where $\theta$ is the angle of the tangent with the absis axis.
Best known for $\beta < \infty$

**Theorem (Lubetzky, Martinelli, Sly, Toninelli ’11)**

For $d = 2$ and $\beta > \beta_c$ and any $\epsilon$ one has

$$T_{\text{mix}}(\epsilon) \leq \exp((\log L)^2).$$

For $d \geq 3$ best bound so far is $\exp(L^{d-2}(\log L)^2)$ [Sugimine ’02].
Zero Temperature

Is easier as what one has to observe is just the evolution of the set of $\ldots$. However, some basic questions are still open in this case.

Set $\mathcal{T}_+ = \{\text{time at which the last } \ldots \text{ disappears}\}$. 
Result for the zero temperature model (d=2,3)

Theorem (L. R. Fontes, R. H. Schonmann, and V. Sidoravicius ’02, Caputo,Martinelli,Simenhaus,Toninelli ’10)

For \( d = 2 \) and \( \beta = \infty \)

\[
cL^2 \leq T_+ \leq CL^2
\]

with probability \( 1 - O(L^{-\infty}) \).

Theorem (Caputo,Martinelli,Simenhaus,Toninelli ’10)

For \( d = 3 \) and \( \beta = \infty \)

\[
L^2(\log L)^{-C} \leq T_+ \leq L^2(\log L)^C,
\]

with probability \( 1 - O(L^{-\infty}) \).
Zero-temperature $d \geq 4$

**Theorem (L ’11)**

For all $d \geq 4$ and $\beta = \infty$

$$\mathcal{T}_+ \leq L^2 (\log L)^C,$$

(7)

with probability $1 - O(L^{-\infty})$.

For the moment, we got no interesting lower bound for $\mathcal{T}_+$
A set of rules

(i) Adding — at any stage of the dynamics only slows it.
(ii) Having non constant boundary condition (instead of all + slows down the dynamic).
(iii) Freezing updates slows the dynamic (Peres and Winkler censoring inequality)
Getting a much weaker result for $d = 4$

**Proposition**

For all $d \geq 4$ and $\beta = \infty$

$$T_+ \leq L^3 (\log L)^C,$$

with probability $1 - O(L^{-\infty})$. 

(8)
Getting a much weaker result for $d = 4$

$T = 0$
Getting a much weaker result for $d = 4$

$$T = 0$$
Getting a much weaker result for $d = 4$

$$T = L^2(\log L)^c$$
Getting a much weaker result for $d = 4$

\[ T = L^2(\log L)^c \]
Getting a much weaker result for $d = 4$

$$T = 2L^2(\log L)^c$$
Getting a much weaker result for $d = 4$

$$T = 2L^2(\log L)^c$$
Getting a much weaker result for $d = 4$

\[ T = 3L^2(\log L)^c \]
Getting a much weaker result for $d = 4$

$$T = 3L^2(\log L)^c$$
Getting the best result

We use the following key proposition of (CMST ’10)

**Proposition**

Consider the dynamic in $3D$ in $B(0, L) \setminus B(0, L - 2(\log L)^{3/2})$ with + boundary condition outside and $-$ boundary condition inside them. There exists a constant $c$ ($c = 10$ works)

$$\mathbb{P} \left[ \exists x \in B(0, L) \setminus B(0, L - 2(\log L)^{3/2}), \sigma_x(L(\log L)^c) = - \right] = O(L^{-\infty})$$
Ingredients for the proof for $d = 3$
(Caputo, Martinelli, Simenhaus, Toninelli ’10)

- Estimate on mixing time for dynamic on monotone surfaces (built on work by Wilson’04).
- Fluctuation results for almost flat monotone surfaces using connection with dimer-covering and random tiling (and various results of Kenyon, Okounkov, Sheffield on dimer models)
\[ T = 0 \]

\[ 2(\log L)^{3/2} \]

\[ (\log L)^{3/2} \]

L
$T = L (\log L)^c$

$2(\log L)^{3/2}$

$(\log L)^{3/2}$

$L$
Strategy of proof 1

\[ 8L(\log L)^{3/2} \]
Strategy of proof 2
Strategy of proof 2

$2(\log L)^{3/2}$
Strategy of proof 2

\[ T = L(\log L)^c \]
Strategy of proof 2

\[ T = L(\log L)^c \]
Strategy of proof 2

\[ T = 2L(\log L)^c \]
Strategy of proof 2

\[ T = 2L(\log L)^c \]
Strategy of proof 2

\[ T = 3L(\log L)^c \]
Strategy of proof 2

\[ T = 3L(\log L)^c \]
Strategy of proof 2

\[ T = 4L \ast L(\log L)^c \]
Strategy of proof 2

CUBE OF SIDELENGTH $L$
Final Remarks

- For $d \geq 5$ result is not built up from the result on dimension $d - 1$ but directly from dimension 3 using a similar construction.
- The result for $d \geq 3$ can not be obtained from the result in dimension 2 because equilibrium fluctuation for interface in dimension 2 are of order $\sqrt{n}$. 