Moderate deviations for random walk in random scenery

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Abstract: We prove annealed moderate deviation principles for the simple random walk in an independent, identically distributed random scenery under the assumption that the scenery satisfies Cramér’s condition. We consider all dimensions $d \geq 2$. An important ingredient of the proofs are concentration inequalities for self-intersection local times of random walks. In $d = 3$ our result relies on recent large deviation estimates of Asselah (2006), and in $d = 2$ on moderate deviation principles for the renormalised self-intersection local times of planar random walks recently obtained by Bass, Chen and Rosen (2006).

1. Introduction

In the world of stochastic processes in random environments, the random walks in random scenery represent a class of processes with fairly weak interaction. Nevertheless, they have deservedly received a lot of attention since their introduction by Kesten and Spitzer [KS79] and, independently, by Borodin [Bo79a, Bo79b]. A major reason for this interest is that in $d \leq 2$ the simple random walk in random scenery exhibits super-diffusive behaviour. However, in dimensions $d \geq 3$, when the underlying random walk visits most sites only once, the behaviour of the random walk in random scenery is diffusive. Here finer features, like large deviation behaviour, have to be studied in order to get an understanding of the interaction of walk and scenery.

Random walks in random scenery are easy to define: Suppose $\{S_n : n \in \mathbb{N}\}$ is a random walk on $\mathbb{Z}^d$ started at the origin $S_0 = 0$, and $\{\xi(z) : z \in \mathbb{Z}^d\}$ are independent, identically distributed nondegenerate random variables, which are independent of the walk. Then the process defined by

$$X_n := \sum_{k=1}^{n} \xi(S_k), \quad \text{for } n \in \mathbb{N}$$

is called random walk in random scenery. In this paper the underlying walk is always a simple, symmetric walk in dimension $d \geq 2$, and the random variables $\xi(z)$ are centred, $\mathbb{E}\xi(z) = 0$, satisfy $\mathbb{E}|\xi(z)|^3 < \infty$ and Cramér’s condition,

$$\mathbb{E}\{e^{\theta \xi(z)}\} < \infty, \quad \text{for some } \theta > 0. \quad (1.1)$$

An alternative representation of the process $\{X_n : n \in \mathbb{N}\}$ is given by

$$X_n = \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z),$$

where $\ell_n(z) = \sum_{k=1}^{n} 1\{S_k = z\}$ is the local time of the random walk at the site $z$.

The early papers by Kesten, Spitzer and Borodin establish central limit theorems for the random walk in random scenery. Indeed, assuming that the scenery has mean zero and variance one, it is shown in [KS79] that, for $d \geq 3$,

$$\frac{X_n}{\sqrt{n}} \overset{n \to \infty}{\longrightarrow} \mathcal{N}(0, 2G(0) - 1)$$
holds, where \( G \) is the Green’s function of the underlying random walk. Bolthausen in [Bo89] extended this to the planar case showing that

\[
\frac{X_n}{\sqrt{n \log n}} \xrightarrow{n \to \infty} \mathcal{N}(0, \pi^{-1}).
\]

Hence, *moderate and large deviation problems* for the random walk in random scenery deal with the asymptotic behaviour of \( \mathbb{P}\{X_n \geq b_n\} \) for \( b_n \gg \sqrt{n} \) if \( d \geq 3 \), and \( b_n \gg \sqrt{n \log n} \) if \( d = 2 \). Let us remark for completeness that Kesten and Spitzer have also established a limit theorem in distribution for \( X_n/n^{3/4} \) with non-Gaussian limits for \( d = 1 \), a case we do not consider in this paper.

Large deviation problems for random walks in random scenery have only recently attracted attention, see [GP02, GHK05, GKS05, As06, AC05, AC06] and also [CP01, AC03, Ca04] where Brownian motions are used in place of random walks. The fascination of this subject stems from the rich behaviour that comes to light when large deviations are investigated. The intricate interplay of the walk with the scenery leads to a large number of different regimes depending on

- the dimension \( d \) of the underlying lattice \( \mathbb{Z}^d \),
- the upper tail behaviour of the scenery,
- the size of the deviation studied,

to name just the most important ones. For example, Asselah and Castell [AC06], restricting attention to dimensions \( d \geq 5 \) and sceneries with superexponential decay of upper tails, have identified five regimes with different large deviation speeds. Heuristically, in each regime the walk and the scenery ‘cooperate’ in a different way to obtain the deviating behaviour. Up to now only one of these regimes has been fully treated, including the discussion of explicit rate functions. This is the *very large deviation* regime discussed (together with a number of boundary cases) by Gantert, König and Shi in [GKS05]. In this regime it is assumed that

\[
\log \mathbb{P}\{\xi(z) > x\} \sim -D x^q \quad \text{as } x \uparrow \infty,
\]

for some \( D > 0 \) and \( q > d/2 \). Then, for any \( n \ll b_n \ll n^{\frac{1+q}{d}} \), as \( n \uparrow \infty \),

\[
\log \mathbb{P}\{X_n > b_n\} \sim K n^{-\frac{2q-d}{d+2}} b_n^{\frac{2q}{d+2}}, \tag{1.2}
\]

where \( K = K(D, q, d) > 0 \) is a constant given explicitly in terms of a variational problem. The underlying strategy is that the random walk contracts to grow at a speed of

\[
\frac{n^{\frac{1+q}{d+2}}}{b_n^{\frac{q}{d+2}}} \ll n^{\frac{1}{2}},
\]

and on the (contracted) range of the walk the scenery adopts values of size \( b_n/n \). The right hand side in (1.2) represents the combined cost of these two deviations.

In the present paper we study a *moderate deviation regime*, providing a full analysis including explicit rate functions. This moderate deviation regime extends from the central limit scaling up to the point where tail conditions stronger than Cramér’s condition would have an impact on the speed of the deviations. Heuristically, our results, which will be described in all detail in the next section, show that in \( d \geq 3 \) the deviation is achieved by a moderate deviation of the scenery without any contribution from the walk. The rates therefore agree with those obtained for fixed walk in a random scenery by Guillotin-Plantard in [GP02]. Crucial ingredient of our proof are *concentration inequalities* for self-intersection local times of random walks, see Proposition 3.1.
In \( d = 2 \), by contrast, the regime splits in two parts. If \( \sqrt{n \log n} \ll b_n \ll \sqrt{n \log n} \) then, again, we only have a contribution from the scenery and the walk exhibits typical behaviour. However, if \( \sqrt{n \log n} \ll b_n \ll n^{3/4} \) the random walk contracts, though in a much more delicate way than in the very large deviation regime: The self-intersection local times
\[
\sum_{1 \leq j < k \leq n} 1\{S_j = S_k\}
\]
of the walk, which normally are of order \( n \log n \) are now increased to be of order \( \sqrt{nb_n} \). At the same time, on the (contracted) range of the walk, the scenery values perform a moderate deviation and take values of size \( b_n/n \). Our results in the case \( d = 2 \) rely on moderate deviation principles for the renormalised self-intersection local times of planar random walks recently obtained by Bass, Chen and Rosen \([BCR06]\).

### 2. Main results

Throughout this paper we assume that the random variables \( \xi(z) \) are centred, satisfy \( \mathbb{E} |\xi(z)|^3 < \infty \) and Cramér’s condition \([11]\). We denote by \( \sigma^2 > 0 \) the variance of \( \xi(z) \). For \( d \geq 3 \) we define the Green’s function of the random walk by
\[
G(x) := \sum_{k=0}^{\infty} \mathbb{P}\{S_k = x\} \quad \text{for } x \in \mathbb{Z}^d.
\]

**Theorem 2.1** (Moderate deviations in dimensions \( d \geq 3 \)).

If \( d \geq 3 \) and \( n^{\frac{1}{2}} \ll b_n \ll n^{\frac{2}{3}} \), then, as \( n \uparrow \infty \),
\[
\log \mathbb{P}\{X_n \geq b_n\} \sim -\frac{b_n^2}{2\sigma^2(2G(0) - 1)}.
\]

**Remark** 1 In this regime the deviation is entirely due to the moderate deviation behaviour of the scenery, whereas the random walk does not contribute and behaves in a typical way. Up to the factor \( 2G(0) - 1 \), the moderate deviation rate agrees with that for a real-valued random walk with step distribution \( \xi(0) \).

**Remark** 2 The deviation speed \( n^{2\beta - 1} \), though not the rate function, in this result was identified by Asselah and Castell \([AC06]\) in \( d \geq 5 \) and by Asselah \([As06]\) in \( d = 3 \), for \( b_n = n^\beta \) with \( 1/2 < \beta \leq 2/3 \), under the additional assumptions that the law of \( \xi \) has a symmetric density which is decreasing on the positive half-axis.

Moving to \( d = 2 \), we define \( \varkappa \) to be the optimal constant in the Gagliardo-Nirenberg inequality,
\[
\varkappa := \inf \{ c : \|f\|_4 \leq c \|\nabla f\|_{1/2} \|f\|^{\frac{1}{2}} \text{ for all } f \in C_c^1(\mathbb{R}^2) \}.
\]

This constant features prominently in large deviation results for intersection local times of Brownian motion and random walk intersection local times, see \([Ch04]\) for further discussion of the Gagliardo-Nirenberg inequality and the associated constant \( \varkappa \).

**Theorem 2.2** (Moderate deviations in dimension \( d = 2 \)).

(i) If \( n^{\frac{3}{2}} \sqrt{\log n} \ll b_n \ll n^{\frac{3}{2}} \log n \), then, as \( n \uparrow \infty \),
\[
\log \mathbb{P}\{X_n \geq b_n\} \sim -\frac{\pi b_n^2}{2\sigma^2 n \log n}.
\]
(ii) If \( n^{\frac{1}{2}} \log n \ll b_n \ll n^\frac{3}{4} \), then, as \( n \uparrow \infty \),
\[
\log \mathbb{P}\{X_n \geq b_n\} \sim -\frac{b_n}{\sigma^2 \sqrt{n} \pi^2}.
\]

(iii) Finally, for every \( a > 0 \),
\[
\log \mathbb{P}\{X_n \geq an^{\frac{1}{2}} \log n\} \sim -I(a) \log n,
\]
where
\[
I(a) = \begin{cases} \frac{\pi a^2}{\sigma^2}, & \text{for } a \leq \frac{\sigma}{\pi} \kappa^2, \\ \frac{a}{\sigma \kappa^2} - \frac{1}{2\pi \kappa^4}, & \text{for } a \geq \frac{\sigma}{\pi} \kappa^2. \end{cases}
\]

Remark 3 In regime (i) the deviation is due to the moderate deviation behaviour of the scenery only, but in regimes (ii) and (iii) there is an additional contraction of the walks to achieve the moderate deviation.

3. Concentration inequalities for self-intersection local times

We define the self-intersection local time of the random walk as
\[
B(n) := \sum_{1 \leq j < k \leq n} 1\{S_j = S_k\}.
\]

The following one-sided concentration inequalities are of independent interest:

**Proposition 3.1** (Concentration inequalities). For any \( \varepsilon > 0 \) there exists \( c = c(\varepsilon) > 0 \) such that,

(a) if \( d > 3 \),
\[
\mathbb{P}\{B(n) - \mathbb{E}B(n) \geq \varepsilon n\} \leq \exp\left\{ - c \frac{n^{\frac{3}{2}}}{(\log n)^2} \right\};
\]

(b) if \( d = 3 \),
\[
\mathbb{P}\{B(n) - \mathbb{E}B(n) \geq \varepsilon n\} \leq \exp\left\{ - cn^{\frac{4}{3}} \right\}.
\]

Proposition 3.1(b) is due to Asselah [As06, Proposition 1.1]. In fact, he proves that, for \( y \) large enough,
\[
\exp\left\{ -cy^{\frac{4}{3}} n^{\frac{1}{3}} \right\} \leq \mathbb{P}\{B(n) \geq yn\} \leq \exp\left\{ -cy^{\frac{4}{3}} n^{\frac{1}{3}} \right\},
\]
and that the main contribution to \( B(n) \) comes from sites which are visited less than \( n^{o(1)} \) times. A careful inspection of his proof shows that ‘\( y \) large enough’ can be taken to mean \( yn \geq \mathbb{E}B(n) + \varepsilon n \), so that Proposition 3.1(b) follows from (3.1). The proof of [As06, Proposition 1.1] is based on a delicate and powerful analysis of the number of sites in \( \mathbb{Z}^d \) visited a certain number of times. This method can in principle be extended to dimensions \( d > 3 \), though in this case it does not give the correct large deviation speed.

In this paper we give an independent and direct proof of Proposition 3.1(a), which entirely avoids the delicate discussion of the number of sites visited a certain number of times. In a similar spirit we provide a simple approach to the concentration inequality in dimension \( d = 3 \), which is much easier than Asselah’s method, but only produces the weaker bound
\[
\mathbb{P}\{B(n) - \mathbb{E}B(n) \geq \varepsilon n\} \leq \exp\left\{ -c \frac{n^{\frac{4}{3}}}{\log n} \right\}.
\]
The impact of the logarithmic factor in the denominator on the right hand side on our main result would be a slight restriction of the result of Theorem 2.1 in $d = 3$ to the regime $n^{1/2} \ll b_n \ll n^{2/3} / \log n$.

3.1 Proof of Proposition 3.1

The proof of Proposition 3.1 requires the following ‘folklore’ lemma about the intersection of two independent random walks $\{S_n : n \geq 0\}$ and $\{S'_n : n \geq 0\}$ with $S_0 = S_0'$. Denote

$$A_n := \sum_{i,j=1}^{n} 1\{S_i = S'_j\}.$$  

Lemma 3.2. There exists a constant $\vartheta > 0$ such that,

(a) if $d > 4$, then $\sup_n \mathbb{E} \exp \{\vartheta A_n^{1/2}\} < \infty$;

(b) if $d = 4$, then $\sup_n \mathbb{E} \exp \{\vartheta \frac{1}{\sqrt{\log n}} A_n^{1/2}\} < \infty$;

(c) if $d = 3$, then $\sup_n \mathbb{E} \exp \{\vartheta n^{-\frac{1}{2}} A_n^{2/3}\} < \infty$.

Throughout the proof we denote by $C$ a generic constant, not depending on $n, m$, which may change its value at every occurrence. Define the partial Green’s functions by

$$G_n(x) := \sum_{k=0}^{n} \mathbb{P}\{S_k = x\}, \quad \text{for } n \in \mathbb{N}.$$  

From the definition of $A_n$ we obtain

$$\mathbb{E}A_n^m \leq m! \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} \sum_{1 \leq k_1, \ldots, k_m \leq n} \mathbb{E} \prod_{l=1}^{m} 1\{S_{j_l} = S'_{k_l}\}$$

$$\leq m! \sum_{\sigma \in \mathfrak{S}_m} \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \sum_{x_1, \ldots, x_m} \mathbb{E} \prod_{l=1}^{m} 1\{S_{j_l} = x_l\} \mathbb{E} \prod_{l=1}^{m} 1\{S'_{k_l} = x_{\sigma(l)}\}$$

$$\leq m! \sum_{\sigma \in \mathfrak{S}_m} \prod_{l=1}^{m} G_n(x_l - x_{l-1})G_n(x_{\sigma(l)} - x_{\sigma(l-1)}),$$

where $\mathfrak{S}_m$ denotes the group of all permutations of $\{1, \ldots, n\}$, and we set $x_0 = x_{\sigma(0)} = 0$ for convenience.

Applying Hölder’s inequality,

$$\mathbb{E}A_n^m \leq (m!)^2 \sum_{x_1, \ldots, x_m} \prod_{l=1}^{m} G_n^2(x_l - x_{l-1}) = (m!)^2 \left( \sum_{x \in \mathbb{Z}^d} G_n^2(x) \right)^m,$$

and, recalling, for example from [La91, (3.4)], that

$$\sum_{z \in \mathbb{Z}^d} G_n^2(z) \sim \begin{cases} C \sqrt{n} & \text{if } d = 3, \\ C \log n & \text{if } d = 4, \\ C & \text{if } d > 4, \end{cases}$$

we obtain,

$$\mathbb{E}A_n^m \leq \begin{cases} (m!)^2 C_m n^{m/2} & \text{if } d = 3, \\ (m!)^2 C_m (\log n)^m & \text{if } d = 4, \\ (m!)^2 C_m & \text{if } d > 4, \end{cases}$$
If \( d > 4 \) this implies \( \mathbb{E}(\sqrt{A_n})^m \leq \sqrt{\mathbb{E}A_n^m} \leq m! C^m \), and the result follows by considering the exponential series. The analogous argument gives the result in \( d = 4 \). In \( d = 3 \) we need an extra argument to complete the proof: We write \( \ell(m,n) = \lceil n/m \rceil + 1 \). Using an inequality of Chen, [Ch04, Theorem 5.1] (with \( p = 2 \) and \( a = m \)), we get, for \( n \geq 3 \),

\[
\sqrt{\mathbb{E}A_n^m} \leq \sum_{k_1 + \ldots + k_m = m} \frac{m!}{k_1! \ldots k_m!} \sqrt{\mathbb{E}A_{\ell(m,n)}^{k_1}} \ldots \sqrt{\mathbb{E}A_{\ell(m,n)}^{k_m}} \leq \sum_{k_1 + \ldots + k_m = m} \frac{m!}{k_1! \ldots k_m!} \left( (k_1!)^2 C^{k_1} \ell(m,n)^{k_1/2} \right) \ldots \left( (k_m!)^2 C^{k_m} \ell(m,n)^{k_m/2} \right) \leq \left( \frac{2m-1}{m} \right) m! C^m \left( \frac{n}{m} \right)^{m/4} \leq (m!)^{3/4} C^m n^{m/4},
\]

and therefore \( \mathbb{E}A_n^m \leq (m!)^{3/2} C^m n^{m/2} \). For \( n \leq m \) we get the same estimate immediately from the trivial inequality \( A_n^m \leq n^{2m} \leq (m!)^{3/2} C^m n^{m/2} \). We thus obtain, for all \( n, m \), that

\[
\mathbb{E}(n^{-\frac{1}{3}} A_n^{2/3})^m \leq m! C^m,
\]

and the result in the case \( d = 3 \) follows by taking the exponential series.

For any \( N \in \mathbb{N} \) we use the classical decomposition

\[
B(2^N) - \mathbb{E}B(2^N) = \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} A_{j,k},
\]

\[
A_{j,k} = \sum_{(2k-1)2^{N-j} \leq \ell \leq (2k-1)2^{N-j} - m < (2k)2^{N-j}} \left( 1\{S_l = S_m\} - \mathbb{P}\{S_l = S_m\} \right).
\]

For fixed \( 1 \leq j \leq N \) the random variables \( A_{j,k} \), for \( k = 1, \ldots, 2^{j-1} \), are independent, identically distributed with the law of \( A_{2^{N-j}} - \mathbb{E}A_{2^{N-j}} \). The next proposition exploits this independence, and the moment results of Lemma 3.2 to give large deviation upper bounds.

**Proposition 3.3** (Large deviation upper bounds). For every \( \varepsilon > 0 \) there exists \( c = c(\varepsilon) > 0 \) such that, for all \( x \geq 2^N/N^2 \),

(a) if \( d > 4 \), then \( \mathbb{P}\left\{ \sum_{k=1}^{2^{j-1}} A_{j,k} \geq \varepsilon x \right\} \leq \exp \left\{ - c \sqrt{x} \right\} \);

(b) if \( d = 4 \), then \( \mathbb{P}\left\{ \sum_{k=1}^{2^{j-1}} A_{j,k} \geq \varepsilon x \right\} \leq \exp \left\{ - c \frac{\sqrt{x}}{N} \right\} \);

(c) if \( d = 3 \), then \( \mathbb{P}\left\{ \sum_{k=1}^{2^{j-1}} A_{j,k} \geq \varepsilon x \right\} \leq \exp \left\{ - c \frac{x^{1/3}}{N^{2/3}} \right\} \).
The proof of this result will be given in the next section. To exploit the result for the completion of the proof of Proposition 3.1, we first focus on \(d > 4\), and use the observation
\[
\mathbb{P}\{B(2^N) - \mathbb{E}B(2^N) \geq \varepsilon x\} = \mathbb{P}\left\{\sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} A_{j,k} \geq \varepsilon x\right\} \leq \sum_{j=1}^{N} \mathbb{P}\left\{\sum_{k=1}^{2^{j-1}} A_{j,k} \geq \varepsilon \frac{x}{N}\right\}
\]
(3.3)

For any \(n \geq 2\) there exists the representation
\[
n = 2^{N_1} + 2^{N_2} + \cdots + 2^{N_l},
\]
where \(N_1 > \cdots > N_l \geq 0\) are integers. Write \(n_0 = 0\) and \(n_i = 2^{N_i} + \cdots + 2^{N_l}\) for \(1 \leq i \leq l\). Denote
\[
B_i := \sum_{n_{i-1} < j < k \leq n_i} 1\{S_j = S_k\}, \quad \text{and} \quad D_i := \sum_{n_{i-1} < j < k \leq n_i} 1\{S_j = S_k\}.
\]
Then \(B(n) = \sum_{i=1}^{l} B_i + \sum_{i=1}^{l-1} D_i\). We thus have
\[
\mathbb{P}\{B(n) - \mathbb{E}B(n) \geq \varepsilon n\} \leq \sum_{i=1}^{l} \mathbb{P}\{B_i - \mathbb{E}B_i \geq \varepsilon \frac{n_i}{2}\} + \sum_{i=1}^{l-1} \mathbb{P}\{D_i \geq \varepsilon \frac{n_i}{2}\}.
\]
Note that \(l \leq \log_2 n\). Equation 3.3 for \(N = N_1\) and \(x = \frac{n}{2l} \geq 2^{N_i}/N_i^2\) gives, for sufficiently large \(n\),
\[
\sum_{i=1}^{l} \mathbb{P}\{B_i - \mathbb{E}B_i \geq \varepsilon \frac{n_i}{2}\} \leq (\log_2 n)^2 \exp\left\{-c \frac{\sqrt{n}}{\log n}\right\}.
\]
(3.4)

As (with \(\equiv\) denoting equality of laws)
\[
D_i \equiv \sum_{j=1}^{2^{N_i} - n_i} \sum_{k=1}^{2^{N_i}} 1\{S_j = S_k\} \leq \sum_{j=1}^{2^{N_i}} \sum_{k=1}^{2^{N_i}} 1\{S_j = S_k\} = A(2^{N_i}),
\]
the second sum can be estimated using Chebyshev’s inequality and Lemma 3.2.
\[
\sum_{i=1}^{l-1} \mathbb{P}\{D_i \geq \varepsilon \frac{n_i}{2}\} \leq (\log_2 n) \sup_k \mathbb{E}\exp\{\theta \sqrt{A(k)}\} \exp\left\{-\theta \frac{\sqrt{n}}{2 \log_2 n}\right\}.
\]
(3.5)
The proof of Proposition 3.3 for \(d > 4\) is completed by combining 3.3 and 3.4, and the proof is analogous for dimensions \(d = 3, 4\) (but recall that for \(d = 3\) this approach only gives 3.2).}

### 3.2 Proof of Proposition 3.3

We give different arguments for the case \(d \geq 4\) and the case \(d = 3\). The argument leading to the proof in the case of high dimensions is based on a fine large deviation upper bound for sums of independent random variables, taken from Nagaev [Na79, Theorem 2.3]. In \(d = 3\) the input from Lemma 3.2 is too weak to make good use of the finesses of this result. Here we use a lemma, taken from Bass, Chen and Rosen [BCR05, Lemma 1], on exponential moments of independent random variables in the Gaussian scaling in conjunction with Chebyshev’s inequality. Throughout the proof \(c\) and \(C\) denote generic constants, which may change their values at each occurrence.

Let us start with the details in the high-dimensional case, \(d \geq 5\). Take a continuously differentiable function \(g\): \((0, \infty) \rightarrow \mathbb{R}\) with non-increasing derivative, such that
(a) \( g'(x) > 2/x \) for all \( x > 0 \),
(b) \( g(x) = \vartheta \sqrt{x} \) for all \( x \geq x_0 \),

where \( \vartheta \) is chosen as in Lemma 3.2. Denote

\[
 b_j(N) := \mathbb{E} \left[ \exp \left\{ g(A_{j,1}) \right\} 1\{A_{j,1} > 0\} \right],
\]

and recall that \( b_j(N) \) is uniformly bounded in \( j \leq N \) and \( N \in \mathbb{N} \). By Theorem 2.3 of [Na79] (with \( \gamma_1 = \gamma_2 = \gamma_3 = 1/3, \gamma = 2/3 \) and \( \delta = 2 \)) we obtain the bound

\[
\mathbb{P} \left\{ \sum_{k=1}^{2^j-1} A_{j,k} \geq \varepsilon x \right\} \leq e^{1/2} \exp \left\{ - \frac{a^2 \varepsilon^2 x}{2(a+1)^2 V_j(N)} \right\} + e^{1/2} \exp \left\{ - \frac{2a \varepsilon x}{3S^{-1} \left( \frac{2a \varepsilon x}{3(a+1) V_j(N)} \right)} \right\} + 2^j b_j(N) e^{1/2} \exp \left\{ - g \left( \frac{2}{3} \varepsilon x \right) \right\} + 2^{j-1} \mathbb{P} \left\{ A_{j,1} \geq \frac{2}{3} \varepsilon x \right\},
\]

where \( V_j(N) \) is the variance of \( A_{j,1} \), the constant \( a \) is the unique solution of the equation \( (u+1) = e^{u-1} \), and \( S^{-1} \) is the inverse function of \( u \mapsto S(u) = e^{-g(u)}g'(u)u^2 \). Recall from [Na79, p.765] that the function \( S \) is strictly decreasing. By Chebyshev’s inequality,

\[
\mathbb{P} \left\{ A_{j,1} \geq x \right\} \leq \left( \sup_{N, j \leq N} b_j(N) \right) e^{-g(x)},
\]

and therefore the two terms in (3.8) are bounded by a constant multiple of

\[
2^N \exp \left\{ - g \left( \frac{2}{3} \varepsilon x \right) \right\}, \quad \text{for all } j \leq N.
\]

Recalling the definition of \( g \) gives an upper bound of

\[
C \exp \left\{ - c \sqrt{x} \right\}, \quad \text{for all } N \in \mathbb{N}.
\]

Further, using the boundedness of \( V_j(N) \) and the fact that \( x \geq 2^N/N^2 \), the term in (3.6) is bounded by a constant multiple of \( \exp \{ -c x/N^2 \} \), and is therefore negligible compared to (3.9).

To show that also the term in (3.7) is negligible, recall that the function \( S \) is strictly decreasing. Hence, the term in (3.7) is bounded by

\[
C \exp \left\{ - c \frac{x}{S^{-1} \left( \frac{2}{3} \varepsilon x \right)} \right\}.
\]

From the definition of the functions \( g \) and \( S \) it is easy to see that

\[
S^{-1} \left( \frac{2}{3} \varepsilon x \right) \leq C (\log N)^2.
\]

This implies that the term in (3.7) is bounded by a constant multiple of \( \exp \{ -c x/(\log N)^2 \} \), and is therefore also negligible compared to (3.9). This completes the proof in \( d \geq 5 \). The result in \( d = 4 \) is a modification of this argument, using the random variable \( (N - j)^{-1} A_{j,k} \) instead of \( A_{j,k} \), and is left to the reader.

In \( d = 3 \) we use a different approach, which is based on the following lemma from [BCR05, Lemma 1], where the proof is attributed to Evarist Giné.
Lemma 3.4 (Giné’s lemma). Let $0 < p \leq 1$ and let $\{Y^j_k : k \geq 1\}$ be a family (indexed by $j$) of sequences of i.i.d. random variables with $\mathbb{E}Y^j_k = 0$ and $\limsup_{\delta \downarrow 0} \sup_j \mathbb{E} \exp \{\delta |Y^j_k|^p\} \leq 1$. Then, for some constant $\lambda > 0$, we have

$$
\sup_{n,j} \mathbb{E} \left\{ \lambda \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y^j_k \right|^p \right\} < \infty.
$$

By Lemma 3.2, we have

$$
\sup_N \mathbb{E} \exp \left\{ \vartheta 2^{-N/3} |A_{2N} - \mathbb{E}A_{2N}|^{2/3} \right\} \leq \sup_N \mathbb{E} \exp \left\{ \vartheta 2^{-N/3} \mathbb{E}A_{2N}^{2/3} \right\} < \infty,
$$

and hence,

$$
\limsup_{\delta \downarrow 0} \sup_N \mathbb{E} \exp \left\{ \vartheta 2^{-N/3} |A_{2N} - \mathbb{E}A_{2N}|^{2/3} \right\} \leq \limsup_{\delta \downarrow 0} \left( \sup_N \mathbb{E} \exp \left\{ \vartheta 2^{-N/3} |A_{2N} - \mathbb{E}A_{2N}|^{2/3} \right\} \right)^{\delta/\vartheta}
$$

$$
\leq \limsup_{\delta \downarrow 0} C^{\delta/\vartheta} = 1.
$$

Hence, by Giné’s lemma applied to the family $\{2^{-N/3} \mathcal{A}_{j,k} : k = 1, \ldots, 2^{j-1}\}$ indexed by $j$, there exists $\lambda > 0$ such that,

$$
\infty > \sup_{N,j \leq N} \mathbb{E} \exp \left\{ \lambda 2^{-j/3} 2^{-(N-j)/3} \left| \sum_{k=1}^{2^{j-1}} \mathcal{A}_{j,k} \right|^{2/3} \right\} = \sup_{N,j \leq N} \mathbb{E} \exp \left\{ \lambda 2^{-N/3} \left| \sum_{k=1}^{2^{j-1}} \mathcal{A}_{j,k} \right|^{2/3} \right\}.
$$

Now we get the desired estimate from Chebyshev’s inequality,

$$
\mathbb{P} \left\{ \sum_{k=1}^{2^{j-1}} \mathcal{A}_{j,k} \geq \varepsilon x \right\} = \mathbb{P} \left\{ 2^{-N/3} \left| \sum_{k=1}^{2^{j-1}} \mathcal{A}_{j,k} \right|^{2/3} \geq \varepsilon x^{2/3} \right\}
$$

$$
\leq \exp \left( - \varepsilon \lambda \frac{2^{j/3}}{2^{N/3}} \right) \mathbb{E} \exp \left\{ \lambda 2^{-N/3} \left| \sum_{k=1}^{2^{j-1}} \mathcal{A}_{j,k} \right|^{2/3} \right\} \leq C \exp \left( - c \frac{2^{j/3}}{2^{N/3}} \right),
$$

using that $x \geq 2^N/N^2$ in the last step. This completes the proof of Proposition 3.3 for $d = 3$. ■

4. Moderate deviations in dimensions $d \geq 3$: Proof of Theorem 2.1

4.1 Proof of the upper bound in Theorem 2.1

We fix $\epsilon > 0$ and let $A := 2G(0) - 1 + 3\epsilon$. Our aim is to show that

$$
\limsup_{n \to \infty} \frac{n}{b_n} \log \mathbb{P} \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} \leq - \frac{1}{2\sigma^2} A.
$$

(4.1)

We note that, for any fixed $\eta > 0$,

$$
\mathbb{P} \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} \leq \mathbb{P} \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n, \max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{\eta m}{b_n}, \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \leq An \right\}
$$

$$
+ \mathbb{P} \left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) \geq \frac{\eta m}{b_n} \right\} + \mathbb{P} \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \geq An \right\}.
$$

(4.2)

To see that the second summand is negligible note that

$$
\mathbb{P} \left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) > x \right\} \leq \mathbb{P} \{ \ell_n(z) > x \text{ for some } z \text{ with } |z| \leq n \} \leq (2n + 1)^d \mathbb{P} \{ \ell_n(0) > x \}.
$$
Moreover, for $1 \ll a_n \ll n$, there exists some $c > 0$ such that,
\[
\mathbb{P}\{\ell_n(0) > a_n\} \leq e^{-ca_n}, \quad \text{for all } n \in \mathbb{N}.
\]
This is easy to show, see [GHK05, Lemma 1.3]. Applying this with $a_n = \eta n/b_n$ gives
\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{P}\left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) > \frac{m}{b_n}\right\} \leq \limsup_{n \to \infty} \frac{dn \log n}{b_n^2} - c \frac{\eta n^2}{b_n^2} = -\infty. \tag{4.3}
\]
To see that the third term in (4.2) is negligible recall the definition of $B(n)$ from the previous section, and note the fact that
\[
\sum_{z \in \mathbb{Z}^d} \ell_n^2(z) = n + 2 \sum_{1 \leq k < \ell \leq n} 1\{S_k = S_l\} = n + 2B(n).
\]
We have $\mathbb{E}B(n) \sim n(G(0) - 1)$ and therefore, for all large $n$,
\[
\mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \geq An\right\} \leq \mathbb{P}\left\{ B(n) - \mathbb{E}B(n) \geq \left(\frac{A-1-\epsilon}{2} - G(0)\right)n\right\} = \mathbb{P}\{ B(n) - \mathbb{E}B(n) \geq \epsilon n \}.
\]
From Proposition 3.1 we know that for $b_n \ll n^{2/3}$, if $d \geq 4$,
\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{P}\{ B(n) - \mathbb{E}B(n) \geq \epsilon n \} \leq \limsup_{n \to \infty} -c \frac{n^{3/2}}{b_n^2 \log n} = -\infty,
\]
and, if $d = 3$,
\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{P}\{ B(n) - \mathbb{E}B(n) \geq \epsilon n \} \leq \limsup_{n \to \infty} -c \frac{n^{4/3}}{b_n^2} = -\infty.
\]
Combining this, we get
\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \geq An\right\} = -\infty. \tag{4.4}
\]
It remains to investigate the first term on the right hand side of (4.2). For this purpose, for the moment fix $\{\ell_n(z) : z \in \mathbb{Z}^d\}$ such that
\[
\max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{m}{b_n} \quad \text{and} \quad \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \leq An,
\]
and just look at probabilities (denoted $\mathbb{P}$ with expectation $\mathbb{E}$) for the i.i.d. variables $\{\xi(z) : z \in \mathbb{Z}^d\}$.
\[
\text{Denote } f(h) := \mathbb{E}e^{h\xi(z)} \quad \text{for all } h < \varepsilon,
\]
which is well-defined for some $\varepsilon > 0$ by Cramér’s condition. Recall that
\[
f(h) = \exp \left\{ \frac{1}{2} h^2 \sigma^2(1 + o(h)) \right\}, \quad \text{as } h \downarrow 0.
\]
In particular, given any $\delta > 0$, we may choose a small $\eta > 0$ such that
\[
f\left( \frac{b_n \ell_n(x)}{\sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right) \leq \exp \left\{ (1 + \delta) \frac{b_n^2 \ell_n^2(x)}{2 \sigma^2 \left( \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \right)^2} \right\}, \tag{4.5}
\]
where we use that \( b_n \ell_n(x)/\sum_{z \in \mathbb{Z}^d} \ell_n(z) \leq \eta \). We can now use Chebyshev’s inequality and independence to get that
\[
P \left\{ \sum_{x \in \mathbb{Z}^d} \ell_n(x) \xi(x) \geq b_n \right\} \leq \prod_{x \in \mathbb{Z}^d} f \left( \frac{b_n \ell_n(x)}{\sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right) \exp \left\{ - \frac{b_n^2}{\sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right\}
\leq \exp \left\{ (1 + \delta) \frac{b_n^2}{2\sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right\} \exp \left\{ - \frac{b_n^2}{\sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right\}
\leq \exp \left\{ - (1 - \delta) \frac{b_n^2}{2\sigma^2 A_n} \right\}.
\]
We can now average over the random walk again, and get from (4.2) together with (4.3) and (4.4), recalling that \( \delta > 0 \) was arbitrary,
\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} \leq - \frac{1}{2\sigma^2 A},
\]
as required in (4.1). This completes the proof.

### 4.2 Proof of the lower bound in Theorem 2.1

We impose ‘typical behaviour’ of the \( \ell^2 \)-norm and the \( \ell^\infty \)-norm of the local times of the random walk. More precisely, fix an arbitrary \( \epsilon > 0 \) and also fix \( \eta > 0 \), which we specify later. We have
\[
P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} \geq P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n, \max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{\eta m}{b_n}, \sum_{z \in \mathbb{Z}^d} \ell_n(z)^2 \leq A n \right\}
= \mathbb{E} \left\{ P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} 1 \{ \max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{\eta m}{b_n}, \sum_{z \in \mathbb{Z}^d} \ell_n(z)^2 \leq A n \} \right\},
\]
where \( A := 2G(0) - 1 + 3\epsilon \) and \( P \) refers to the probability with respect to the scenery only. To study the inner probability we now suppose that, for the moment, a random walk sample with local times \( \{ \ell_n(z) : z \in \mathbb{Z}^d \} \) is fixed, such that
\[
\max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{\eta m}{b_n} \quad \text{and} \quad \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \leq A n.
\]
Denote \( \gamma := \mathbb{E}[|\xi(z)|^3] < \infty \), and recall that \( \sigma^2 = \mathbb{E}[|\xi(z)|^2] > 0 \). Hence the variance of the random variable \( \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \) with respect to \( P \) is given by \( \mathbb{V}_n^2 := \sigma^2 \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \) and the Lyapunov ratio by \( L_n := \gamma \mathbb{V}_n^{-3} \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \). By [Na02, Theorem 2] there exist constants \( c_1, c_2 > 0 \) such that, for all \( \sum_{z \in \mathbb{Z}^d} \ell_n(z)^2 \leq 196L_n \),
\[
P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq x \right\} \geq (1 - \Phi(x/\sqrt{\mathbb{V}_n})) \exp \left\{ -c_1 x^3 \mathbb{V}_n^{-3} \right\} (1 - c_2 x L_n \mathbb{V}_n^{-1}),
\]
where \( \Phi \) is the Gaussian error function. Now suppose that \( \eta \) is chosen to satisfy
\[
\eta < \sigma^4/(196\gamma), \quad c_1 \eta \gamma \sigma^{-6} < \epsilon, \quad \text{and} \quad c_2 \eta \gamma \sigma^{-4} < \epsilon.
\]
Using the upper bound on the maximum of the local times, we get that \( L_n \leq \frac{\eta m}{\sigma^2 b_n} \mathbb{V}_n^{-1} \). Therefore,
\[
P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq x \right\} \geq (1 - \Phi(x/\sqrt{\mathbb{V}_n})) \exp \left\{ -c_1 \eta \frac{x^3}{\mathbb{V}_n} \right\} \left(1 - c_2 \gamma \eta \frac{x}{\sigma^2 b_n} \mathbb{V}_n^{-2} \right),
\]
for all $\frac{3}{4} V_n \leq x \leq \frac{b_n V_n}{196 \eta n}$. We can use this inequality for $x = b_n$. Indeed, as $V_n^2 \leq A \sigma^2 n$ we get $b_n \geq \frac{3}{4} V_n$, if $n$ exceeds some constant depending only on $\sigma^2$. Also $V_n^2 \geq \sigma^2 n$ and $\eta < \sigma^2/(196\gamma)$, therefore

$$b_n \leq b_n \sigma^2 V_n^2 / (196 \gamma n) \leq V_n / (196 L_n).$$

Hence,

$$\mathbb{P}\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \} \geq (1 - \Phi \left( \frac{b_n}{\sqrt{V_n}} \right)) \exp \left\{ - c_1 \eta \gamma^2 \frac{\sigma^4}{n} b_n^2 \right\} (1 - c_2 \gamma \sigma^4 \eta). \quad (4.8)$$

Substituting (4.8) into (4.6) gives

$$\mathbb{P}\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \} \geq (1 - c_2 \gamma \sigma^4 \eta) \exp \left\{ - c_1 \gamma \sigma^2 \eta \frac{b_n^2}{n} \right\} \mathbb{E}\left[ (1 - \Phi \left( \frac{b_n}{\sqrt{V_n}} \right)) 1\{ V_n^2 \leq A \sigma^2 n, \max_{z \in \mathbb{Z}^d} \ell_n(z) \leq \frac{2m}{b_n} \} \right] \quad (4.9)$$

$$\geq (1 - \epsilon) \exp \left\{ - \epsilon \frac{b_n^2}{n} \right\} \mathbb{E}\left[ (1 - \Phi \left( \frac{b_n}{\sqrt{V_n}} \right)) 1\{ V_n^2 \leq A \sigma^2 n \} \right] - \mathbb{P}\left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) \geq \frac{2m}{b_n} \right\}.$$

Since, by a standard estimate of the Gaussian error function, $(1 - \Phi(z)) \geq \exp\{- (1 + \eta) z^2 / 2\}$ for all sufficiently large $z$, we get

$$\mathbb{E}\left[ (1 - \Phi \left( \frac{b_n}{\sqrt{V_n}} \right)) 1\{ V_n^2 \leq A \sigma^2 n \} \right] \geq \mathbb{E}\exp \left\{ - \frac{(1 + \eta)b_n^2}{2V_n^2} \right\} - \mathbb{P}\{ V_n^2 \geq A \sigma^2 n \}. \quad (4.10)$$

Using Jensen’s inequality, we obtain

$$\mathbb{E}\exp \left\{ - \frac{(1 + \eta)b_n^2}{2V_n^2} \right\} \geq \exp \left\{ - \frac{(1 + \eta)b_n^2}{2\sigma^2 n} \mathbb{E}\sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \right\}.$$

Recalling, e.g. from [BS95], that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) = 2G(0) - 1$$

almost surely,

and using that $n / \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) \leq 1$, we get that

$$\lim_{n \to \infty} \mathbb{E}\sum_{z \in \mathbb{Z}^d} \ell_n^2(z) = \frac{1}{2G(0) - 1}.$$

Then, for all $n$ sufficiently large,

$$\mathbb{E}\left[ (1 - \Phi \left( \frac{b_n}{\sqrt{V_n}} \right)) \right] \geq \exp \left\{ - \frac{(1 + 2\eta)b_n^2}{2\sigma^2 n (2G(0) - 1 - \epsilon)} \right\}. \quad (4.11)$$

Combining (4.9), (4.10) and (4.11) gives

$$\mathbb{P}\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \} \geq (1 - \epsilon) \exp \left\{ - \frac{(1 + 2\eta)}{2\sigma^2 (2G(0) - 1 - \epsilon)} \frac{b_n^2}{n} \right\}$$

$$- \mathbb{P}\left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) \geq \frac{2m}{b_n} \right\} - \mathbb{P}\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z)^2 \geq A n \}.$$

The required lower bound follows from the estimates (4.3) and (4.4) for the subtracted probabilities, and the fact that $\epsilon > 0$ can be chosen arbitrarily small, whence $\eta$ also becomes arbitrarily small.
5. Moderate deviations in dimension \(d = 2\): Proof of Theorem 2.2

We use the following moderate deviations principle for the self-intersection local time in the planar case, which is due to Bass, Chen and Rosen [2006, Theorem 1.1]: If \(x_n \to \infty\) and \(x_n = o(n)\), then for every \(\lambda > 0\),

\[
\lim_{n \to \infty} \frac{1}{x_n} \log \mathbb{P}\{ B(n) - \mathbb{E}B(n) \geq \lambda n x_n \} = -\lambda x^{-4}, \tag{5.1}
\]

where again \(x\) is the optimal constant in the Gagliardo-Nirenberg inequality.

5.1 Proof of Theorem 2.2 (i)

The proof is largely analogous to that of Theorem 2.1 replacing Proposition 3.1 by (5.1). Starting with the upper bound, for any fixed \(\epsilon > 0\), we use the decomposition

\[
\mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^2} \ell_n(z) \xi(z) \geq b_n \right\} \leq \mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^2} \ell_n(z) \xi(z) \geq b_n, \max_{z \in \mathbb{Z}^2} \ell_n(z) \leq \frac{\sqrt{\log n}}{b_n}, \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \leq A n \log n \right\}
+ \mathbb{P}\left\{ \max_{z \in \mathbb{Z}^2} \ell_n(z) \geq \frac{\sqrt{\log n}}{b_n} \right\} + \mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \geq A n \log n \right\},
\]

where \(A = \pi^{-1} + 4\epsilon\). The estimate for the last probability follows from (5.1). Indeed, from \(n + 2B(n) = \sum \ell_n^2(z)\) and \(\mathbb{E}B(n) \sim (2\pi)^{-1} n \log n\) we get, for sufficiently large \(n\),

\[
\mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \geq A n \log n \right\} \leq \mathbb{P}\left\{ B(n) - \mathbb{E}B(n) \geq \frac{1}{2} (A - \pi^{-1} - \epsilon) n \log n \right\} \leq n^{-\epsilon x^{-4}},
\]

hence, as \(b_n \ll n^x \log n\),

\[
\lim_{n \to \infty} \frac{n \log n}{b_n^2} \log \mathbb{P}\left\{ \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \geq A n \log n \right\} = -\infty. \tag{5.2}
\]

Moreover, as in (5.3), we get

\[
\limsup_{n \to \infty} \frac{n \log n}{b_n^2} \log \mathbb{P}\left\{ \max_{z \in \mathbb{Z}^2} \ell_n(z) > b_n^{-1} \sqrt{n (\log n)} \right\}
\leq \limsup_{n \to \infty} \frac{2n (\log n)^2}{b_n^2} - c \frac{n^2 (\log n)^4}{b_n^4} = -\infty. \tag{5.3}
\]

We now look at fixed local times \(\{\ell_n(z): z \in \mathbb{Z}^2\}\) satisfying the conditions \(\max \ell_n(z) \leq b_n^{-1} \sqrt{n (\log n)}\) and \(\sum \ell_n^2(z) \leq A n \log n\). Note that, together with the trivial inequality \(\sum \ell_n^2(z) \geq n\), this implies

\[
\lim_{n \to \infty} \frac{b_n \ell_n(z)}{\sigma^2 \sum \ell_n^2(z)} = 0.
\]

Hence, for arbitrary \(\delta > 0\), if \(n\) is sufficiently large, an application of Chebyshev’s inequality and the estimate (5.5) for the Laplace transform \(f(\xi(z))\), gives, for \(n\) larger than an absolute constant,

\[
\mathbb{P}\left\{ \sum_{x \in \mathbb{Z}^d} \ell_n(x) \xi(x) \geq b_n \right\} \leq \prod_{x \in \mathbb{Z}^d} f\left( \frac{b_n \ell_n(x)}{\sigma^2 \sum z \in \mathbb{Z}^2 \ell_n^2(z)} \right) \exp \left\{ - \frac{b_n^2}{2 \sigma^2 \sum z \in \mathbb{Z}^2 \ell_n^2(z)} \right\}
\leq \exp \left\{ - (1 - \delta) \frac{b_n^2}{2 \sigma^2 A n \log n} \right\}.
\]
Averaging over the local times again, we obtain
\[
\limsup_{n \to \infty} \frac{n \log n}{b_n} \log P \left\{ \sum_{z \in \mathbb{Z}^2} \ell_n(z)\xi(z) \geq b_n, \ \max_{z \in \mathbb{Z}^2} \ell_n(z) \leq \frac{\sqrt{n}(\log n)^2}{b_n}, \ \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \leq A \log n \right\} \leq \frac{(1-\delta)}{2\sigma^2 A},
\]
so that the claimed upper bound follows, as \( \epsilon, \delta > 0 \) were arbitrary.

Turning to the lower bound, we fix \( \epsilon > 0 \) again, and use that
\[
P \left\{ \sum_{z \in \mathbb{Z}^2} \ell_n(z)\xi(z) \geq b_n \right\} \geq E \left\{ \sum_{z \in \mathbb{Z}^2} \ell_n(z) \xi(z) \geq b_n \right\} \times 1 \left\{ \max_{z \in \mathbb{Z}^2} \ell_n(z) \leq \frac{\sqrt{n}(\log n)^2}{b_n}, \ \sum_{z \in \mathbb{Z}^2} \ell_n^2(z) \leq A \log n \right\},
\]
where \( A = \pi^{-1} + 4\epsilon \). To obtain a lower bound for the inner probability we argue as in Theorem 2.1 relying on the estimates of [Na02, Theorem 2]. This gives
\[
P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \right\} \geq (1 - \Phi(\frac{\pi}{n})) \exp \{ -c_1 \gamma \sigma^{-6} b_n^2 n^{-\frac{3}{2}} (\log n)^3 \} \left( 1 - c_2 \gamma \sigma^{-4} n^{-\frac{1}{2}} (\log n)^4 \right).
\]
From [BS95, Theorem 1.2] we recall that the sequence
\[
Y_n := \frac{1}{n} \sum_{z \in \mathbb{Z}^d} \ell_n^2(z) - \frac{1}{\pi} \log n
\]
of random variables is uniformly integrable and convergent in distribution. Observe that
\[
\frac{n \log n}{\sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} = \pi - \frac{\pi Y_n}{\frac{1}{n} \sum_{z \in \mathbb{Z}^d} \ell_n^2(z)}
\]
and the subtracted fraction on the right converges in probability to zero and is dominated by \( \pi |Y_n| \) and is hence uniformly integrable. Therefore
\[
\lim_{n \to \infty} E \left[ \frac{n \log n}{\sum_{z \in \mathbb{Z}^d} \ell_n^2(z)} \right] = \pi.
\]
Repeating the arguments of the \( d \geq 3 \) case gives,
\[
P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \right\} \geq (1 - c_2 \gamma \sigma^{-4} n^{-\frac{1}{2}} (\log n)^4) \exp \{ -c_1 \gamma \sigma^{-6} b_n^2 n^{-\frac{3}{2}} (\log n)^3 \} \exp \{ -\frac{(1+\epsilon)^2 \pi b_n^2}{2n^2 \log n} \} \exp \{ -\frac{(1+\epsilon)^2 \pi b_n^2}{2n^2 \log n} \} \geq P \left\{ \max_{z \in \mathbb{Z}^d} \ell_n(z) \geq \frac{\sqrt{n}(\log n)^2}{b_n} \right\} - P \left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \geq A \log n \right\}.
\]
The result follows, by observing that the first two factors on the right converge to one, recalling (5.3), (5.4) and noting that as \( \epsilon > 0 \) was arbitrary.

**5.2 Proof of Theorem 2.2(ii)**

Again, we start with the upper bound. Since \( E B(n) \sim (2\pi)^{-1} n \log n \), we can conclude from (5.1) that, for \( \log n \ll x_n \ll n \),
\[
\lim_{n \to \infty} \frac{1}{x_n} \log P \{ B(n) \geq \lambda nx_n \} = -\lambda x_n^{-4}.
\]
For arbitrary $N \geq 1$ and $\delta > 0$,
\[
\mathbb{P}\{X_n \geq b_n\} \leq \sum_{i=0}^{N-1} \mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \in (i\delta a_n, (i+1)\delta a_n]\right\} + \mathbb{P}\{\max \ell_n(z) > \gamma_n\} + \mathbb{P}\{B(n) > N\delta a_n\},
\]
(5.7)
where $\gamma_n := \eta n \log n/b_n$ and $a_n = b_n n^{1/2}$. Note that $a_n \gg n \log n$. Hence, in view of (5.6),
\[
\mathbb{P}\{B(n) > N\delta a_n\} \leq \exp\left\{-\frac{N\delta a_n}{2\sigma^2 n}\right\}
\]
(5.8)
for all sufficiently large $n$. On the other hand, using [GHK05, Lemma 1.3] as in the other cases,
\[
\mathbb{P}\left\{\max \ell_n(z) > \gamma_n\right\} \leq \exp\left\{-c\frac{n \log n}{b_n \log b_n}\right\}.
\]
(5.9)
Fix any $i \geq 1$. It is easy to check that on the event $\{\max \ell_n(z) \leq \gamma_n, B(n) \in (i\delta a_n, (i+1)\delta a_n]\}$ we can use Chebyshev’s inequality as in all previous cases, which gives
\[
\mathbb{P}\left\{\sum_{z \in \mathbb{Z}^2} \ell_n(z) \xi(z) \geq b_n\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n^2}{2\sigma^2 \sum \ell_n^2(z)}\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n^2}{4\sigma^2 (i+1)\delta a_n}\right\},
\]
where in the last step we used the relation $\sum \ell_n^2(z) \sim 2B(n)$. Therefore,
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \in (i\delta a_n, (i+1)\delta a_n]\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n}{4\sigma^2 (i+1)\delta a_n}\right\} \mathbb{P}\{B(n) > i\delta a_n\}.
\]
Applying (5.6) again, and recalling the definition of $a_n$, we get, for all sufficiently large $n$,
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \in (i\delta a_n, (i+1)\delta a_n]\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n}{4\sigma^2 (i+1)\delta a_n^{1/2}} + \frac{(1 - \epsilon) i \delta b_n}{\varepsilon^4 n^{1/2}}\right\}.
\]
(5.10)
It remains to consider
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \leq \delta a_n\right\}.
\]
Observe that this probability is bounded by
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, \mathbb{E}B(n)/2 \leq B(n) \leq \delta a_n\right\} + \mathbb{P}\{B(n) < \mathbb{E}B(n)/2\}.
\]
Applying again Chebyshev’s inequality to the first term, we have
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \leq \delta a_n\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n}{4\sigma^2 \delta a_n^{1/2}}\right\} + \mathbb{P}\{B(n) < \mathbb{E}B(n)/2\}.
\]
By Theorem 1.2 of [BCR05b] with $b_n = n$ and $\theta = 1/2$,
\[
\mathbb{P}\{B(n) < \mathbb{E}B(n)/2\} \leq \exp\{-cn^{1/2}\}.
\]
Therefore,
\[
\mathbb{P}\left\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) \leq \delta a_n\right\} \leq \exp\left\{-\frac{(1 - \epsilon) b_n}{4\sigma^2 \delta a_n^{1/2}}\right\} + \exp\{-cn^{1/2}\}.
\]
(5.11)
Combining (5.7) – (5.11) gives us
\[
\mathbb{P}\{X_n \geq b_n\} \leq \sum_{i=0}^{N-1} \exp\left\{ -\frac{(1-\epsilon)b_n}{4\sigma^2(i+1)\delta n^{1/2}} - \frac{(1-\epsilon)i\delta b_n}{\chi^4 n^{1/2}} \right\} + \exp\{-cn^{1/2}\} + \exp\left\{ -\frac{c}{b_n \log b_n} n \log n \right\} + \exp\left\{ -\frac{N\delta b_n}{2\chi^4 n^{1/2}} \right\}.
\] (5.12)

It is easily seen, that
\[
\lim_{n \to \infty} \frac{n^{1/2}}{b_n} \log \sum_{i=0}^{N-1} \exp\left\{ -\frac{(1-\epsilon)b_n}{4\sigma^2(i+1)\delta n^{1/2}} - \frac{(1-\epsilon)i\delta b_n}{\chi^4 n^{1/2}} \right\} = -(1-\epsilon) \min_{0 \leq i \leq N-1} \left( \frac{1}{4\sigma^2(i+1)\delta} + \frac{i\delta}{\chi^4} \right).
\]

Furthermore, we can choose \(\delta\) and \(N\) such that
\[
\min_{0 \leq i \leq N-1} \left( \frac{1}{4\sigma^2(i+1)\delta} + \frac{i\delta}{\chi^4} \right) \geq (1-\epsilon) \min_{x > 0} \left( \frac{1}{4\sigma^2 x} + \frac{x}{\chi^4} \right) = (1-\epsilon) \frac{1}{\sigma \chi^2}.
\]

Therefore, for all \(n\) large enough,
\[
\sum_{i=0}^{N-1} \exp\left\{ -\frac{(1-\epsilon)b_n}{4\sigma^2(i+1)\delta n^{1/2}} - \frac{(1-\epsilon)i\delta b_n}{\chi^4 n^{1/2}} \right\} \leq \exp\left\{ -(1-\epsilon)^3 \frac{b_n}{\sigma \chi^2 n^{1/2}} \right\}.
\] (5.13)

Evidently,
\[
\exp\{-cn^{1/2}\} + \exp\left\{ -\frac{c}{b_n \log b_n} n \log n \right\} = o\left( \exp\left\{ -(1-\epsilon)^3 \frac{b_n}{\sigma \chi^2 n^{1/2}} \right\} \right).
\] (5.14)

for \(n^{1/2}\log n \ll b_n \ll n^{3/4}\). On the other hand, for all large \(N\),
\[
\exp\left\{ -\frac{N\delta b_n}{2\chi^4 n^{1/2}} \right\} = o\left( \exp\left\{ -(1-\epsilon)^3 \frac{b_n}{\sigma \chi^2 n^{1/2}} \right\} \right).
\] (5.15)

Substituting (5.13) – (5.15) into (5.12) and taking into account that \(\epsilon > 0\) was arbitrary, we have
\[
\lim_{n \to \infty} \frac{n^{1/2}}{b_n} \log \mathbb{P}\{X_n \geq b_n\} \leq -\frac{1}{\sigma \chi^2}.
\]

To obtain a lower bound, note that for all \(0 < \mu < \lambda\),
\[
\mathbb{P}\{X_n \geq b_n\} \geq \mathbb{P}\{X_n \geq b_n, B(n) \in [\mu a_n, \lambda a_n], \max \ell_n(z) \leq \gamma_n\}.
\] (5.16)

Evidently, \(L_n \leq n^{-2} \gamma_n V_n^{-1} = \gamma^{-2} \eta n \log n V_n^{-1} b_n^{-1}\) on the set \(\{\max \ell_n(z) \leq \gamma_n\}\). Therefore, from (1.7) we obtain the bound
\[
P\left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq x \right\} \geq (1 - \Phi(x/V_n)) \exp\left\{ -c_1 \gamma^{-2} \eta \frac{x^3 n \log n}{b_n V_n^4} \right\} \left( 1 - c_2 \gamma^{-2} \eta \frac{xn \log n}{b_n V_n^2} \right)
\]

for \(3V_n/2 \leq x \leq b_n \sigma^2 V_n^2/(196 \eta \gamma \log n)\). If \(B(n) > \mu a_n\), then \(V_n^2 > 2 \sigma^2 \mu a_n\). Thus, we can continue with
\[
\geq \left( 1 - \Phi\left( x/\sqrt{2 \sigma^2 \mu b_n n^{1/2}} \right) \right) \exp\left\{ -c_1 \gamma^{-6} \eta \frac{x^3 \log n}{4 \mu^2 b_n^2} \right\} \left( 1 - c_2 \gamma^{-4} \eta \frac{xn^{1/2} \log n}{2 \mu b_n} \right).
\]

For all large \(n\), we have \(3V_n/2 \leq b_n \leq b_n \sigma^2 V_n^2/(196 \eta \gamma \log n)\) on the set \(\{B(n) \in [\mu a_n, \lambda a_n]\}\). Hence
\[
P\left\{ \sum_{z \in \mathbb{Z}^d} \ell_n(z) \xi(z) \geq b_n \right\} \geq \exp\left\{ -(1 + \epsilon) \frac{b_n}{4 \sigma^2 n^{1/2}} - \frac{c_1 B_n \log n}{4 \mu^2 b_n^6} \right\} \left( 1 - c_2 \gamma n^{-1/2} \log n / 2 \mu \sigma^2 b_n \right).
\]
Therefore,
\[
\mathbb{P}\{X_n \geq b_n\} \geq \exp\left\{-(1+\epsilon)\frac{b_n}{4\mu \sigma^2 n^{1/2}} - \frac{c_1 \gamma \eta \log n}{4\mu^2 \sigma^6} \right\}\left(1 - \frac{c_2 \gamma n^{1/2} \log n}{2\sigma^4 \mu b_n} \right) \\
\times \left[\mathbb{P}\{B(n) \in [\mu a_n, \lambda a_n]\} - \mathbb{P}\{\max \ell_n(z) > \gamma_n\}\right].
\]  
(5.17)

From (5.16) we conclude that for all \(\mu < \lambda\),
\[
\log \mathbb{P}\{B(n) \in [\mu a_n, \lambda a_n]\} \sim -\frac{\mu b_n}{\sigma^2 n^{1/2}}.
\]  
(5.18)

Applying (5.18) and (5.7) to the right hand side of (5.17), we get for \(n^{1/2} \log n \ll b_n \ll n^{3/4}\) the bound
\[
\liminf_{n \to \infty} \frac{n^{1/2}}{b_n} \log \mathbb{P}\{X_n \geq b_n\} \geq \frac{1+\epsilon}{4\mu \sigma^2} - \frac{\mu}{\sigma^4}.
\]
Since \(\epsilon\) and \(\mu\) are arbitrary,
\[
\liminf_{n \to \infty} \frac{n^{1/2}}{b_n} \log \mathbb{P}\{X_n \geq b_n\} \geq \min_{\mu > 0} \left(\frac{1}{4\mu \sigma^2} + \frac{\mu}{\sigma^4}\right) = -\frac{1}{\sigma \kappa^2}.
\]
This completes the proof of Theorem 2.2 (ii).

5.3 Proof of Theorem 2.2 (iii)
We now assume that \(b_n = a\sqrt{n \log n}\). In this case we use the following decomposition,
\[
\mathbb{P}\{X_n \geq b_n\} \leq \mathbb{P}\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) - \mathbb{E}B(n) \leq \delta a_n\} \\
+ \sum_{i=1}^{\n} \mathbb{P}\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) - \mathbb{E}B(n) \in (i\delta a_n, (i+1)\delta a_n]\} \\
+ \mathbb{P}\{\max \ell_n(z) > \gamma_n\} + \mathbb{P}\{B(n) - \mathbb{E}B(n) > N\delta a_n\},
\]
here \(a_n = n \log n, \gamma_n = \eta n \log n/b_n\). Estimating every term as in the proof of the upper bound in (ii) and using the relation \(\mathbb{E}B(n) \sim (2\pi)^{-1} n \log n\), one can get
\[
\limsup_{n \to \infty} \frac{1}{\log n} \mathbb{P}\{X_n \geq b_n\} \leq -\min_{x \geq 0} \frac{a^2}{4\sigma^2((2\pi)^{-1} + x)} - \frac{x}{\sigma^4} = I(a).
\]
In order to get a lower bound we consider the cases \(a \leq \sigma/\pi \kappa^2\) and \(a > \sigma/\pi \kappa^2\) separately. In the first case we use
\[
\mathbb{P}\{X_n \geq b_n\} \geq \mathbb{P}\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, |B(n) - \mathbb{E}B(n)| \leq \delta a_n\},
\]
and in the second case
\[
\mathbb{P}\{X_n \geq b_n\} \geq \mathbb{P}\{X_n \geq b_n, \max \ell_n(z) \leq \gamma_n, B(n) - \mathbb{E}B(n) \in (\mu a_n, \lambda a_n]\}
\]
for some \(0 < \mu < \lambda\). The further proof is similar to that of the lower bound in Theorem 2.2 (ii) and details are left to the reader.

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