New Complexity Bounds for Cylindrical Decompositions of Sub-Pfaffian Sets

Savvas Pericleous
Nicolai Vorobjov

Abstract

Tarski-Seidenberg principle plays a key role in real algebraic geometry and its applications. It is also constructive and some efficient quantifier elimination algorithms appeared recently. However, the principle is wrong for first-order theories involving certain real analytic functions (e.g., an exponential function). In this case a weaker statement is sometimes true, a possibility to eliminate *one sort* of quantifiers (either $\forall$ or $\exists$). We construct an algorithm for a cylindrical cell decomposition of a closed cube $I^n \subset \mathbb{R}^n$ compatible with a semianalytic subset $S \subset I^n$, defined by analytic functions from a certain broad finitely defined class (Pfaffian functions), modulo an oracle for deciding emptiness of such sets. In particular the algorithm is able to eliminate one sort of quantifiers from a first-order formula. The complexity of the algorithm and the bounds on the output are doubly exponential in $O(n^2)$.

1 Introduction

Semianalytic sets are defined as subsets of points in $\mathbb{R}^n$ satisfying Boolean combinations of atomic formulae of the kind $f > 0$, where f's are real analytic functions defined in a common open domain $G \subset \mathbb{R}^n$. Subanalytic sets are defined as images of relatively proper real analytic maps of semianalytic sets. If functions f are polynomials, then these two classes of sets coincide (Tarski-Seidenberg principle). An equivalent formulation of this statement is that the first-order theory of reals admits quantifier elimination. It plays a key role in many aspects and applications of real algebraic geometry. However, Tarski-Seidenberg principle is wrong already if one of atomic f's is an exponential function, in which case a subanalytic set may not be semianalytic [16]. Thus, the quantifier elimination is not generally possible in a first-order theory with real analytic functions. A theorem due to Gabrielov [6, 7] shows however that at least one sort of quantifiers (either $\forall$ or $\exists$) *can* be eliminated. This is equivalent to saying that the complement to a subanalytic set is subanalytic.

It is well-known that Tarski-Seidenberg principle is constructive. Original Tarski's proof provided an algorithm for quantifier elimination (com-

puting of a projection) with a non-elementary complexity. In mid-70-s, [3] and [20] proposed elementary algorithms (doubly-exponential in the number of variables n). In recent years the problem had received a significant attention, had attracted a number of powerful mathematical techniques, and as a result some very efficient quantifier elimination algorithms were designed (see [1, 11, 12, 17]).

Attempting to extend the complexity results from algebraic to real analytic case, we have firstly to restrict the class of real analytic functions to a finitely defined subclass which would include as many as possible important analytic functions (for example, all algebraic functions, exponentials, logarithms, etc.), and for whose members a natural concept of a *size* or *format* would be definable. A suitable class of such kind is formed by *Pfaffian* functions. Pfaffian functions are solutions of triangular systems of first order partial differential equations with polynomial coefficients. *Semi-Pfaffian* sets, defined by systems of equations and inequalities between these functions, are characterized by global finiteness properties [13, 14] (formal definitions are given in Section 2). This means that their basic geometric and topological characteristics can be explicitly estimated in terms of formats of their defining formulae. *Sub-Pfaffian* sets are relatively proper images of semi-Pfaffian sets, and their complements are also sub-Pfaffian.

A common technique for proving quantifier elimination results is constructing a *cylindrical cell decomposition* of the set defined by the quantifier-free part of a given formula (see definition in the next section), i.e. representing this set as a disjoint union of geometrically simple *cells*, homeomorphic to open balls of some dimensions, which induces (via projections) similar decompositions on a certain filtration of subspaces. This method was used in [3,20] to obtain doubly-exponential upper complexity bounds for algebraic case (more efficient modern algorithms [1, 11, 12, 17] don't use cylindrical decomposition).

The technique of cylindrical cell decomposition was applied to Pfaffian case in the context of model-theoretic study of *o-minimality* (see [5,19]). The complexity estimates which can be extracted from these works are apparently non-elementary.

Recently Gabrielov and Vorobjov in [9] suggested an algorithm which produces cylindrical cell decompositions of sub-Pfaffian sets in $\mathbb{R}^n$. In particular, this algorithm finds complements to sub-Pfaffian sets, in other words eliminates one sort of quantifiers from prenex first-order formulae involving Pfaffian functions. As a model of computation [9] uses a *real numbers machine* (Blum-Shub-Smale model) [2] equipped with an *oracle* for deciding the feasibility of any system of Pfaffian equations and inequalities. The complexity bound of this algorithm, the number and formats of cells are doubly exponential in $O(n^3)$ (assuming that each oracle call has a unit cost).

In the present paper we obtain a new upper complexity bound by using a very different and more elementary technique. As in [9], this bound is doubly exponential in the number of variables and, being formally incomparable with

the one from [9], is better for a long Pfaffian chain for defining functions. We rely on the two known results: the Khovanskii's upper bound on the number of connected components of a semi-Pfaffian set [13, 14] and Gabrielov's algorithm for computing the closure of a semi-Pfaffian set [8]. Unlike [9], we do not use a stratification algorithm.

2 Pfaffian functions and sub-Pfaffian sets

Definition 2.1. (See [13,14] , and [10].) A *Pfaffian chain* of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of real analytic functions $f_1, \ldots, f_r$ in G satisfying Pfaffian equations

$$df_j(X) = \sum_{1 \leq i \leq n} g_{ij}(X, f_1(X), \ldots, f_j(X))dX_i$$

for $1 \leq j \leq r$. Here $g_{ij}(X, Y)$ are polynomials in $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_j)$ of degree not exceeding α. A function

$$f(X) = P(X, f_1(X), \ldots, f_r(X))$$

where $P(X, Y_1, \ldots, Y_r)$ is a polynomial of degree not exceeding $\beta \geq 1$ is a *Pfaffian function* of order r and degree (α, β).

Example 2.2.

1. *Pfaffian functions of order 0 and degree* $(1, \beta)$ *are polynomials of degree not exceeding* β.
2. *The exponential univariate function* $f(X) = e^{aX}$ *is a Pfaffian function of order 1 and degree* $(1, 1)$ *in* $\mathbb{R}$, *due to the equation* $df(X) = af(X)dX$.
3. *The function* $f(X) = 1/X$ *is a Pfaffian function of order 1 and degree* $(2, 1)$ *in the domain* $X \neq 0$, *due to the equation* $df(X) = -f^2(X)dX$.
4. *The logarithmic function* $f(X) = \ln(|X|)$ *is a Pfaffian function of order 2 and degree* $(2, 1)$ *in the domain* $X \neq 0$, *due to the equations* $df(X) = g(X)dX$ *and* $dg(X) = -g^2(X)dX$ *with* $g(X) = 1/X$.

For more examples of Pfaffian functions see [10, 14].

Lemma 2.3. *(See [10, 14])*

1. *The sum (resp. product) of two Pfaffian functions,* f_1 *and* f_2, *of orders* r_1 *and* r_2 *and degrees* (α_1, β_1) *and* (α_2, β_2), *is a Pfaffian function of the order* $r_1 + r_2$ *and degree* $(max(\alpha_1, \alpha_2), max(\beta_1, \beta_2))$ *(resp.* $(max(\alpha_1, \alpha_2), \beta_1 + \beta_2)$*). If the two Pfaffian functions are defined by the same Pfaffian chain of order* r, *then the order of the sum and product is also* r.

2. *A partial derivative of a Pfaffian function of order r and degree (α, β) is a Pfaffian function of order r and degree $(\alpha, \alpha + \beta - 1)$.*

In what follows we only consider the "restricted" case in which Pfaffian functions are defined also on the boundary of the domain.

Let $I^k = [0,1]^k$ denote the unit cube in $\mathbb{R}^k$.

Definition 2.4. (*Semi- and sub-Pfaffian set.*)

1. A set $S \subset \mathbb{R}^s$ is called *semi-Pfaffian* in an open domain $G \subset \mathbb{R}^s$ if it consists of points from G satisfying a Boolean combination of atomic equations and inequalities $f = 0, g > 0$, where f, g are Pfaffian functions having a common Pfaffian chain defined in the domain G.

2. Consider $I^{m+n} \subset G$, where $G \subset \mathbb{R}^{m+n}$ is an open domain, and the projection map $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$. A subset $W \subset \mathbb{R}^n$ is called *(restricted) sub-Pfaffian* if $W = \pi(S)$ for semi-Pfaffian set $S \subset I^{m+n}$.

According to [6, 7], the complement $I^n \setminus W$ in $I^n = \pi(I^{n+m})$ of a sub-Pfaffian set W is also sub-Pfaffian.

Definition 2.5. (*Format.*) For a semi-Pfaffian set

$$S = \bigcup_{1 \le l \le M} \{f_l = 0, g_{l1} > 0, \ldots, g_{lJ_l} > 0\} \subset G \subset \mathbb{R}^s, \tag{1}$$

where f_i, g_{ij} are Pfaffian functions with a common Pfaffian chain, of order r and degree (α, β), defined in an open domain G, its *format* is a quintuple (N, α, β, r, s), where $N = 1 + \sum_{1 \le l \le M}(J_l + 1)$. Let $D = \alpha + \beta$. For $s = m + n$ and a sub-Pfaffian set $W \subset \mathbb{R}^n$ such that $W = \pi(S)$, its *format* is the format of S. In the sequel we will use the notation $g_l > 0$ for the system of inequalities $g_{l1} > 0, \ldots, g_{lJ_l} > 0$.

Proposition 2.6. *([13,14]) The number of connected components of a semi-Pfaffian set S with the format (N, α, β, r, s), does not exceed*

$$2^{r^2} s^{O(r)} (ND)^{O(r+s)}. \tag{2}$$

Corollary 2.7. *The number of connected components of a sub-Pfaffian set $W = \pi(S)$, with format (N, α, β, r, s), defined by a formula having only existential quantifiers, does not exceed bound (2).*

As a model of computation we use a *real numbers machine* (Blum-Shub-Smale model) [2] equipped with an *oracle* for deciding the feasibility of any system of Pfaffian equations and inequalities. An oracle is a subroutine which can be used by the algorithm any time the latter needs to check feasibility. We assume that this procedure always gives the correct answer though we do not specify how it actually works. For some classes of Pfaffian functions the feasibility problem is decidable on real numbers machines or Turing machines

with explicit (singly-exponential) complexity bounds. Apart from polynomials, such class form, for example, terms of the kind $P(e^h, X_1, \ldots, X_n)$ where h is a fixed polynomial in $X_1, \ldots, X_n$ and P is an arbitrary polynomial in $X_0, X_1, \ldots, X_n$ (see [18]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in the usual sense. As far as the computational complexity is concerned, we assume that each oracle call has the unit cost.

Definition 2.8. The *closure* $cl(S)$ of a sub-Pfaffian set S in an open domain G is an intersection with G of the usual topological closure of S:

$$cl(S) = \{x \in G \,:\, \forall \varepsilon > 0\, \exists z \in S\, (|x - z| < \varepsilon)\}.$$

The *frontier* ∂S of S is $cl(S) \setminus S$.

Lemma 2.9. *Let S be a semi-Pfaffian set in an open domain $G \subset \mathbb{R}^s$, of format (N, α, β, r, s), defined by (1) where $s = n + m$ and the variables are $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_m)$. There is an algorithm which produces a Boolean formula $F(X, Y)$ in a disjunctive normal form with atomic Pfaffian functions such that for any fixed $y \in \mathbb{R}^m$ the closure $cl(S \cap \{Y = y\}) \subset \mathbb{R}^n$ coincides with $\{F(X, y)\}$. The format of $\{F(X, Y)\}$ is*

$$((Nd)^{O((s+r)s)}, \alpha, d^{O(s)}, r, s),$$

where $d = 2^{r^2}(sD)^{s+r}$ and $D = \alpha + \beta$. The complexity of the algorithm does not exceed $(Nd)^{O((s+r)s)}$.

Proof. The proof of this lemma is a straightforward parameterization of the proof of Theorem 1.1 from [8]. □

If a set S is defined by a formula Ψ, then $F(X, Y)$ from the proof of Lemma 2.9 will be sometimes denoted by $cl(\Psi)$.

Definition 2.10. ([5, 19]) Cylindrical cell is defined as follows.

1. Cylindrical 0-cell in $\mathbb{R}^n$ is an isolated point.
2. Cylindrical 1-cell in $\mathbb{R}$ is an open interval $(a, b) \subset \mathbb{R}$.
3. For $n \geq 2$ and $0 \leq k < n$, a cylindrical $(k+1)$-cell in $\mathbb{R}^n$ is either a *section over* C, i.e., a graph of a continuous bounded function $f : C \to \mathbb{R}$ where C is a cylindrical $(k+1)$-cell in $\mathbb{R}^{n-1}$ equipped with coordinates $X_2, \ldots, X_n$, or else a *sector over* C, i.e., a set of the form

$$(f, g) \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_2, \ldots, x_n) \in C \text{ and}$$

$$f(x_2, \ldots, x_n) < x_1 < g(x_2, \ldots, x_n)\}$$

where C is a cylindrical k-cell in $\mathbb{R}^{n-1}$ and $f, g : C \longrightarrow \mathbb{R}$ are continuous bounded functions satisfying $f(x_2, \ldots, x_n) < g(x_2, \ldots, x_n)$ for all points $(x_2, \ldots x_n) \in C$.

Clearly, a cylindrical k-cell is homeomorphic to an open k-dimensional ball, and its closure is homeomorphic to a closed k-dimensional ball.

Definition 2.11. Cylindrical cell decomposition, say $\mathcal{D}$, of a subset $A \subset \mathbb{R}^n$ is defined as follows.

1. If $n = 1$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is A.
2. If $n \geq 2$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells in $\mathbb{R}^n$ whose union is A and there is a cell decomposition $\mathcal{D}'$ of $\pi(A)$ such that for each cell C of $\mathcal{D}$, the set $\pi(C)$ is a cell of $\mathcal{D}'$, where $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is the projection map onto the coordinate subspace of $X_2, \ldots, X_n$. We say that $\mathcal{D}'$ is induced by $\mathcal{D}$.

Definition 2.12. If $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$ and $\mathcal{D}$ is a cylindrical cell decomposition of A, then $\mathcal{D}$ is *compatible* with B if for all $C \in \mathcal{D}$ either $C \subset B$ or $C \cap B = \emptyset$ (i.e. some $\mathcal{D}' \subset \mathcal{D}$ is a cylindrical cell decomposition of $B \cap A$).

3 The main result

We describe an algorithm for producing a cylindrical cell decomposition of a semi-Pfaffian set S in the closed unit cube $I^n = [0,1]^n \subset \mathbb{R}^n$. By the definition, this decomposition induces a cylindrical decomposition of the projection W of S onto a subspace $\mathbb{R}^m$, $m \leq n$.

More precisely, an input of the algorithm is a semi-Pfaffian set S defined by (1) with $s = n$, and we assume that S is contained in $I^n \subset G$.

Let

$$\pi : \mathbb{R}^n \to \mathbb{R}^m,$$
$$\pi : (X_1, \ldots, X_n) \mapsto (X_{n-m+1}, \ldots, X_n)$$

be the projection map with $\pi(S) = W$.

The output of the algorithm is a cylindrical cell decomposition $\mathcal{D}_n$ of I^n compatible with S. Each cell is described by a formula of the type

$$\pi'\Big(\bigcup_{1 \leq i \leq M'} \bigcap_{1 \leq j \leq M''} \{h_{ij} *_{ij} 0\}\Big),$$

where h_{ij} are Pfaffian functions in $n' \geq n$ variables, π' is the projection map $\pi' : \mathbb{R}^{n'} \to \mathbb{R}^n$, $*_{ij} \in \{=, <, >\}$, and M', M'' are certain integers. By the definition of a cylindrical cell decomposition, $\mathcal{D}_n$ induces a cylindrical decomposition $\mathcal{D}_m$ of the cube $I^m = \pi(I^n) \subset \mathbb{R}^m$ compatible with W. Using an oracle the algorithm can then decide which cells from $\mathcal{D}_m$ belong to W and which to its complement $I^m \setminus W$.

We prove that the number of cells in $\mathcal{D}_n$, the components of the format and the complexity of the algorithm are less than

$$(\alpha + \beta N)^{r^{O(n)} 2^{O(n^2)}}.$$

Note that in [9] the bound for these parameters is

$$N^{(r+m+n)^{O(d)}}(\alpha+\beta)^{r^{O(d(m+dn))}},$$

where $d = \dim(W) \leq n$. Note that if S is *semialgebraic*, then our bound is essentially the same as the best known upper bound in a cylindrical cell decomposition for the polynomial case [3,20]. Recall also result of Davenport and Heintz [4], that real quantifier elimination is doubly exponential (and hence any cylindrical algebraic decomposition algorithm should have the same complexity).

4 Description of a cell decomposition

Let $S \subset I^n \subset G \subset \mathbb{R}^n$ be a semi-Pfaffian set, defined by (1) with $s = n$ and having format (N, α, β, r, n).

Firstly, we reduce the formula defining set S to a simple special form which is essentially a single Pfaffian equation. Introduce a new variable X_0 and the function

$$f \equiv \prod_{1\leq i\leq M}\Big(\big(f_i^2+(X_0-iN)^2\big)\cdot\prod_{1\leq j\leq J_i}\big(g_{ij}^2+(X_0-iN-j)^2\big)\Big).$$

Notice that f is a Pfaffian function of order r and degree $(\alpha, 2\beta N)$.

Let $\mathcal{D}$ be a cylindrical decomposition of $I^n \times [0, N^2] \subset \mathbb{R}^{n+1}$ compatible with $\{f = 0\} \cap (I^n \times [0, N^2])$, and $\mathcal{D}'$ be the cylindrical decomposition of I^n induced by $\mathcal{D}$.

By the definition of the cylindrical decomposition, $\mathcal{D}'$ is compatible with $\pi(\{f = 0\})$ where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map onto the subspace $\{X_0 = 0\}$. Generally, $\pi(\{f = 0\}) \neq S$.

Lemma 4.1. *The cylindrical decomposition $\mathcal{D}'$ is compatible with S.*

Proof. We need to prove that for any cell C' of $\mathcal{D}'$ either $C' \subset S$ or $C' \cap S = \emptyset$. Suppose that contrary to this, for a cell C' of $\mathcal{D}'$, there are points $x, y \in C'$ such that $x \in \{f_i = 0, g_{i1} > 0, \ldots, g_{iJ_i} > 0\}$ for a certain i, and $y \notin S$. In particular, $y \notin \{f_i = 0, g_{i1} > 0, \ldots, g_{iJ_i} > 0\}$, i.e., either $g_{ij}(y) \leq 0$ for some $1 \leq j \leq J_i$ or $f_i(y) \neq 0$. In the case $g_{ij}(y) \leq 0$, since C' is connected and $g_{ij}(x) > 0$, there is a point $z \in C' \cap \{g_{ij} = 0\}$ and therefore a point $(z, iN + j) \in \{f = 0\}$. The point $(z, iN + j)$ belongs to a cell, say C, of $\mathcal{D}$. Note that $\pi(C) = C'$. Clearly, $C \subset \{g_{ij} = 0\} \cap \{X_0 = iN + j\}$. It follows that $C' \subset \{g_{ij} = 0\}$ which contradicts to $x \in C'$ and $g_{ij}(x) > 0$. In the case of $f_i(y) \neq 0$, the point $(y, iN) \notin \{f = 0\}$. The point (y, iN) belongs to a cell, say C'', of $\mathcal{D}$, and $\pi(C'') = C'$. But $C'' \cap \{f = 0\} = \emptyset$, since $\mathcal{D}$ is compatible with $\{f = 0\}$, which is a contradiction, since $x \in C'$ and $f_i(x) = 0$. □

We proved that it is sufficient to construct a cylindrical decomposition of the intersection $\{f = 0\} \cap (I^n \times [0, N^2])$. For simplicity of notations, in

what follows we assume that the function f has just n variables $X_1, \ldots, X_n$ and $S = \{f = 0\} \cap I^n$. In the remaining part of this section we give a non-constructive description of a certain cylindrical cell decomposition of I^n compatible with $\{f = 0\}$.

Define:

- $I^k_m(a) \equiv I^k \cap \{X_m = a\}$, $1 \leq m \leq k$, $a \in [0,1]$.
- $I^k_m \equiv I^k_m(0) \cup I^k_m(1)$, $1 \leq m \leq k$.
- $L^k_0(a) \equiv I^k$ and $L^k_m(a) \equiv \bigcap_{1 \leq i \leq m} I^k_i(a)$, $1 \leq m \leq k$, $a \in [0,1]$.
- $L^k_m \equiv \bigcap_{1 \leq i \leq m} I^k_i$, $1 \leq m \leq k$.

Without loss of generality assume that $\{f = 0\} \cap I^n_1 = \emptyset$. Set $V \equiv (\{f = 0\} \cap I^n) \cup I^n_1$. Note that the format of V is $(O(n2^n), \alpha, 2\beta N, r, n)$.

The description of a cell decomposition proceeds by induction on n.

Definition 4.2. For a subanalytic curve Γ (a subanalytic set of dimension at most 1) in $\mathbb{R}^n$, define:

1. $E_k(\Gamma)$ to be the set of all points of local extrema of X_k-coordinate on $cl(\Gamma)$.
2. $R(\Gamma)$ to be the set of all ramification points of $cl(\Gamma)$. (A point x is called a ramification point of $cl(\Gamma)$, if for all sufficiently small $0 < t \in \mathbb{R}$ the intersection of $cl(\Gamma)$ with a sphere of the radius t centered at x, contains at least three different points.)
3. $B(\Gamma) \equiv \partial\Gamma = cl(\Gamma) \setminus \Gamma$.
4. The set of all *special* points relative to X_k-coordinate to be $S_k(\Gamma) \equiv E_k(\Gamma) \cup R(\Gamma) \cup B(\Gamma)$.

Observe that an isolated point in Γ is a special point.

We make several initial steps of the induction.

Let $n = 1$. Then let $\Omega^{(0)}_s = S_1(V)$ and define $\Omega^{(0)}_0 = \Omega^{(0)}_s$. For all pairs of points $x, y \in \Omega^{(0)}_s$ consider the set $V \cap \{1/2(x+y)\}$ and denote by $\Omega^{(0)}_m$ the union of all these sets. Notice that if $\{f = 0\}$ is finite, then $\Omega^{(0)}_m = \Omega^{(0)}_0$. Each member of $\Omega^{(0)}_0$ is a zero-dimensional cylindrical cell. A cylindrical cell decomposition $\mathcal{D}$ of I^1 compatible with V and therefore with $\{f = 0\} \cap I^1$ consists of these points and open intervals on the line between them. One can enumerate alternatively these points and intervals by successive non-negative integers j_1 in the ascending along X_1 order by assigning index $j_1 = 0$ to 0, index $j_1 = 1$ to its neighbouring interval, and so on. Notice that $|\mathcal{D}| < 2|\Omega^{(0)}_0|$.

Let $n = 2$. Then for every fixed value ω of X_2-coordinate define finite sets $\Omega_0^{(0)}(\omega)$ and $\Omega_m^{(0)}(\omega)$ as in case $n = 1$ by restricting V to $\{X_2 = \omega\}$. Let

$$\Gamma_0^{(1)} \equiv \bigcup_{\omega \in L_1^2(0)} \Omega_0^{(0)}(\omega), \quad \Gamma_m^{(1)} \equiv \bigcup_{\omega \in L_1^2(0)} \Omega_m^{(0)}(\omega).$$

Clearly, $\Gamma_0^{(1)}$, $\Gamma_m^{(1)}$ are 1-dimensional (not necessarily closed) subsets of I^2. Observe that $L_1^2 \subset \Gamma_0^{(1)} \subset \Gamma_m^{(1)} \subset V$.

Let

$$\Omega_s^{(1)} = S_2(\Gamma_0^{(1)}) \cup S_2(\Gamma_m^{(1)}).$$

For all $x = (x_1, x_2) \in \Omega_s^{(1)}$ denote by $\Omega_0^{(1)}$ the union of finite sets of the kind $\Gamma_0^{(1)} \cap \{X_2 = x_2\}$. For all pairs of points $x = (x_1, x_2), y = (y_1, y_2) \in \Omega_s^{(1)}$ denote by $\Omega_m^{(1)}$ the union of finite sets of the kind $\Gamma_m^{(1)} \cap \{X_2 = 1/2(x_2 + y_2)\}$.

Let $\omega_1 < \omega_2$ be two neighbouring X_2-coordinates of points from $\Omega_0^{(1)}$ (that is, there are no X_2-coordinates ω of points from $\Omega_0^{(1)}$ such that $\omega_1 < \omega < \omega_2$). Then for each $\omega \in (\omega_1, \omega_2)$, the set $\Omega_0^{(0)}(\omega) \subset \{X_2 = \omega\}$ consists of the same finite number of points. Let us enumerate these points and intervals between them, as we did in case $n = 1$, by successive non-negative integers in the ascending along X_1 order. It is clear that the set of all points having the same index for all $\omega \in (\omega_1, \omega_2)$ is an open interval of the curve $\Gamma_0^{(1)}$, which is a one-dimensional cylindrical cell being a graph of a continuous function defined on an interval in the 1-dimensional set $L_1^2(0)$. The set of all intervals having the same index for all $\omega \in (\omega_1, \omega_2)$ is an open 2-dimensional cylindrical cell being the set of points strictly between the non-intersecting graphs of two continuous functions defined on an interval in $L_1^2(0)$.

Now we can describe all zero-, one-, and two-dimensional cells of the cylindrical decomposition of I^2 that is compatible with V. Enumerate each cell by a 2-multi-index (j_1, j_2) in a following way. Index j_2 enumerates (by successive non-negative integers starting from zero) alternatively points in $\Omega_0^{(1)} \cap L_1^2(0)$ and intervals between these points in $L_1^2(0)$ in the ascending along X_2 order. For a fixed value of j_2, index j_1 enumerates points in $\Omega_0^{(0)}(\omega) \subset \{X_2 = \omega\}$ and intervals between them (as in case $n = 1$), where ω is either the X_2-coordinate of the point in $\Omega_0^{(1)} \cap L_1^2(0)$ having index j_2, or the X_2-coordinate of a point in the interval between two neighbouring points of $\Omega_0^{(1)} \cap L_1^2(0)$ having index j_2.

It is easy to see that the defined family of the cylindrical cells is a cylindrical cell decomposition of I^2 compatible with V and therefore with $\{f = 0\} \cap I^2$. A cell having index (i, j) is cylindrical over the cell with index $(0, j)$ that belongs in the decomposition of $L_1^2(0)$. Observe that the number of cells in this decomposition is $O(|\Omega_0^{(1)}|)$.

We proceed to the description of a general induction step.

For every fixed value ω of X_n-coordinate finite sets of points of the kind $\Omega_0^{(n-2)}(\omega)$ and $\Omega_m^{(n-2)}(\omega)$ can be defined by applying the inductive hypothesis

to $V \cap \{X_n = \omega\}$. An important property of these sets, is that there are formulae (with quantifiers) $\Phi_0^{(n-2)}(X_1, \ldots, X_{n-1}, X_n)$ and $\Phi_m^{(n-2)}(X_1, \ldots, X_{n-1}, X_n)$ having free variables $X_1, \ldots, X_n$ and not depending on ω, such that the replacement of the variable X_n by ω gives formulae $\Phi_0^{(n-2)}(X_1, \ldots, X_{n-1}, \omega)$ and $\Phi_m^{(n-2)}(X_1, \ldots, X_{n-1}, \omega)$ in free variables $X_1, \ldots, X_{n-1}$ defining the sets $\Omega_0^{(n-2)}(\omega)$ and $\Omega_m^{(n-2)}(\omega)$ respectively for the section $\{X_n = \omega\}$. Let

$$\Gamma_0^{(n-1)} \equiv \{\Phi_0^{(n-2)}(X_1, \ldots, X_{n-1}, X_n)\}, \quad \Gamma_m^{(n-1)} \equiv \{\Phi_m^{(n-2)}(X_1, \ldots, X_{n-1}, X_n)\}.$$

Clearly, $\Gamma_0^{(n-1)}$, $\Gamma_m^{(n-1)}$ are 1-dimensional (not necessarily closed) subsets of I^n.

Observe that $L_{n-1}^n \subset \Gamma_0^{(n-1)} \subset \Gamma_m^{(n-1)} \subset V$. Moreover, for any $k = 2, \ldots, n-1$ by the definitions of $\Gamma_*^{(n-1)}$, where $* \in \{0, m\}$, we have the inclusions $\pi_k(\Gamma_*^{(n-1)}) \subset \Gamma_*^{(n-1)}$, where π_k denotes the projection on the subspace of coordinates $X_k, X_{k+1} \ldots, X_n$.

Let

$$\Omega_s^{(n-1)} = S_n(\Gamma_0^{(n-1)}) \cup S_n(\Gamma_m^{(n-1)}).$$

For all points $x = (x_1, \ldots, x_n) \in \Omega_s^{(n-1)}$ denote by $\Omega_0^{(n-1)}$ the union of finite sets of the kind $\Gamma_0^{(n-1)} \cap \{X_n = x_n\}$. For all pairs of points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \Omega_s^{(n-1)}$ denote by $\Omega_m^{(n-1)}$ the union of finite sets of the kind $\Gamma_m^{(n-1)} \cap \{X_n = 1/2(x_n + y_n)\}$.

Let the index j_n enumerate in the ascending along X_n order alternatively points in $\Omega_0^{(n-1)} \cap L_{n-1}^n(0)$ and intervals between these points on $L_{n-1}^n(0)$. Let $\omega_1 < \omega_2$ be two neighbouring X_n-coordinates of points from $\Omega_0^{(n-1)}$.

Assume that the interval (ω_1, ω_2) is indexed by j_n and a point $\omega \in (\omega_1, \omega_2)$. It follows from the inductive hypothesis that there is a certain cylindrical cell decomposition of the intersection $I_n^n(\omega) = I^n \cap \{X_n = \omega\}$ compatible with $V \cap \{X_n = \omega\}$ and all cells are enumerated by $(n-1)$-multi-indices. In the next section we will prove that for all $\omega \in (\omega_1, \omega_2)$ the sets of multi-indices coincide. Moreover, we will prove that any fixed multi-index corresponds to cells of the same dimension and finally that the union of all p-cells for $p = 0, 1, \ldots, n-1$ having the same multi-index $(j_1, \ldots, j_{n-1})$ for all $\omega \in (\omega_1, \omega_2)$ is a cylindrical $(p+1)$-cell to which we will assign multi-index $(j_1, \ldots, j_{n-1}, j_n)$.

Let ω be the X_n-coordinate of the point in $\Omega_0^{(n-1)} \cap L_{n-1}^n(0)$ having index j_n. By the inductive hypothesis there is a cylindrical cell decomposition of $I_n^n(\omega)$. All cells of this decomposition are also the elements of a cell decomposition of I^n. If a cell in $I_n^n(\omega)$ has a multi-index $(j_1, \ldots, j_{n-1})$, then considering it as a cell in I^n we assign to it the multi-index $(j_1, \ldots, j_{n-1}, j_n)$.

We prove in the next section that the described decomposition $\mathcal{D}$ is compatible with V and therefore with $\{f = 0\} \cap I^n$. Observe that its total number of cells is $O(|\Omega_0^{(n-1)}|)$.

5 Cell decomposition is well defined

Let $\omega_1 < \omega_2$ be two neighbouring X_n-coordinates of points from $\Omega_s^{(n-1)}$ and $\alpha_1,\ \alpha_2$ be any two numbers such that $\omega_1 < \alpha_1 < \alpha_2 < \omega_2$. According to the inductive hypothesis (of the induction described in previous section), on both $I_n^n(\alpha_1)$ and $I_n^n(\alpha_2)$ certain cylindrical cell decompositions compatible with V are defined. Let $(i_1, \ldots, i_{n-1})$ be the multi-index of a cylindrical p-cell C_1 in $I_n^n(\alpha_1)$.

Lemma 5.1. *There exists a cylindrical p-cell C_2 in $I_n^n(\alpha_2)$ having the same multi-index $(i_1, \ldots, i_{n-1})$.*

Proof. According to the definition of ω_1, ω_2, the set $\Omega_s^{(n-1)} \cap \{\omega_1 < X_n < \omega_2\} = \emptyset$. It follows that

$$\Gamma_m^{(n-1)} \cap \{\alpha_1 \le X_n \le \alpha_2\}$$

consists of a finite number of disjoint curve segments each of which is homeomorphic to $[0, 1]$, and on each of which X_n monotonously increases. Therefore, the sets of points of the kind $\Omega_*^{(n-2)}(\alpha_1)$ for $I_n^n(\alpha_1)$ and $\Omega_*^{(n-2)}(\alpha_2)$ for $I_n^n(\alpha_2)$, where $* \in \{0, m\}$, are in a natural bijective correspondence: $x \in \Omega_*^{(n-2)}(\alpha_1)$ corresponds to $y \in \Omega_*^{(n-2)}(\alpha_2)$ if and only if x and y belong to the same connected component of the intersection $\Gamma_*^{(n-1)} \cap \{\alpha_1 \le X_n \le \alpha_2\}$.

Let $x^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})$, $x^{(2)} = (x_1^{(2)}, \ldots, x_n^{(2)}) \in \Omega_*^{(n-2)}(\alpha_1)$, and j, $1 \le j \le n-1$, be such that

(1) $x_n^{(1)} = x_n^{(2)} = \alpha_1, x_{n-1}^{(1)} = x_{n-1}^{(2)}, \ldots, x_{j+1}^{(1)} = x_{j+1}^{(2)}$ and $x_j^{(1)} < x_j^{(2)}$,

(2) in the interval $(x_j^{(1)}, x_j^{(2)})$ there is no j-coordinate of any point from $\Omega_*^{(j-1)}$ with respect to $\{X_n = x_n^{(1)}, \ldots, X_{j+1} = x_{j+1}^{(1)}\}$.

There exist certain $y^{(1)} = (y_1^{(1)}, \ldots, y_n^{(1)})$, $y^{(2)} = (y_1^{(2)}, \ldots, y_n^{(2)}) \in \Omega_*^{(n-2)}(\alpha_2)$ which are the images of $x^{(1)}, x^{(2)}$ respectively under the bijective correspondence described above.

To prove the lemma, it is sufficient to show that $y^{(1)}, y^{(2)}$ satisfy the conditions similar to (1), (2) for $x^{(1)}, x^{(2)}$. Namely:

(i) $y_n^{(1)} = y_n^{(2)} = \alpha_2$, $y_{n-1}^{(1)} = y_{n-1}^{(2)}, \ldots, y_{j+1}^{(1)} = y_{j+1}^{(2)}$ and $y_j^{(1)} < y_j^{(2)}$;

(ii) in the interval $(y_j^{(1)}, y_j^{(2)})$ there is no j-coordinate of any point from $\Omega_*^{(j-1)}$ with respect to $\{X_n = y_n^{(1)}, \ldots, X_{j+1} = y_{j+1}^{(1)}\}$.

Indeed, suppose that (i), (ii) are established. Index i_{n-1}, by definition, enumerates either the X_{n-1}-projection of a point $x \in \Omega_0^{(n-2)}(\alpha_1)$ or an interval between two neighbouring X_{n-1}-projections of some points $x^{(1)}, x^{(2)} \in$

$\Omega_0^{(n-2)}(\alpha_1)$. In the first case, according to the properties (i), (ii), the image point $y \in I_n^n(\alpha_2)$ of x has the same last index i_{n-1}. In the second case, by the same argument, the interval between points $y^{(1)}, y^{(2)}$ has number i_{n-1}. Repeating this argument by induction for indices $i_{n-2}, \ldots, i_1$ we prove that there exists a p-cell on $I_n^n(\alpha_2)$ with multi-index $(i_1, \ldots, i_{n-1})$.

Now we proceed to the proof of conditions (i), (ii). Let $\Gamma_{x^{(1)},y^{(1)}}$ (respectively, $\Gamma_{x^{(2)},y^{(2)}}$) be the connected component of the intersection $\Gamma_*^{(n-1)} \cap \{\alpha_1 \leq X_n \leq \alpha_2\}$ realizing the bijection between $x^{(1)}, y^{(1)}$ (respectively, between $x^{(2)}, y^{(2)}$). Suppose that (i) is false because there exists $k \in \{j+1, \ldots, n-1\}$ such that $y_k^{(1)} \neq y_k^{(2)}$. Let $s \in \{j+1, \ldots, n-1\}$ be the maximum among such numbers k and $y_s^{(1)} < y_s^{(2)}$. According to the construction of the set $\Gamma_*^{(n-1)}$, we have that $\pi_s(\Gamma_*^{(n-1)}) \subset \Gamma_*^{(n-1)}$, where π_s denotes the projection onto the subspace $\{X_1 = \cdots = X_{s-1} = 0\}$ equipped with coordinates $X_s, X_{s+1} \ldots, X_n$. In particular, $\pi_s(\Omega_*^{(n-2)}(\alpha)) \subset \Omega_*^{(n-2)}(\alpha) \subset I_n^n(\alpha)$, where $\alpha = \alpha_1$ or α_2.

Define the points $y^{(3)} = \pi_s(y^{(1)}), y^{(4)} = \pi_s(y^{(2)})$ which belong to

$$\Omega_*^{(n-2)}(\alpha_2) \cap \{X_{s+1} = y_{s+1}^{(1)}, \ldots, X_n = y_n^{(1)}\}$$

and the point $x^{(3)} = \pi_s(x^{(1)})$, which belongs to

$$\Omega_*^{(n-2)}(\alpha_1) \cap \{X_{s+1} = x_{s+1}^{(1)}, \ldots, X_n = x_n^{(1)}\}.$$

Thus, $y_i^{(3)} = y_i^{(4)} = x_i^{(3)} = 0$ for every $1 \leq i < s$, $y_s^{(3)} = y_s^{(1)}$, $y_s^{(4)} = y_s^{(2)}$ (so, $y_s^{(3)} < y_s^{(4)}$) and $x_s^{(3)} = x_s^{(1)} = x_s^{(2)}$, since by assumption $\pi_s(x^{(1)}) = \pi_s(x^{(2)})$.

Define curves $\Gamma' = \pi_s(\Gamma_{x^{(1)},y^{(1)}})$ and $\Gamma'' = \pi_s(\Gamma_{x^{(2)},y^{(2)}})$ which both are connected and contained in $\Gamma_*^{(n-1)}$. We have: $y^{(3)} \neq y^{(4)}$, $y^{(3)} \in \Gamma'$, $y^{(4)} \in \Gamma''$, and $x^{(3)} \in \Gamma' \cap \Gamma''$. It follows that $x^{(3)} \in S_n(\Gamma_*^{(n-1)})$, which contradicts to the choice of α_1, α_2.

We have proved that for all $k \in \{j+1, \ldots, n\}$ the equality $y_k^{(1)} = y_k^{(2)}$ is true. The inequality $y_j^{(1)} < y_j^{(2)}$ (see (i)) can be proved by a symmetric argument. The property (ii) can be proved similarly. □

Let C_1, C_2 be p-cells in $I_n^n(\alpha_1)$ and $I_n^n(\alpha_2)$ respectively, having the same multi-index $(i_1, \ldots, i_{n-1})$. By inductive hypothesis for any $j = 1, 2$, either $C_j \subset V$ or $C_j \cap V = \emptyset$.

Lemma 5.2. *$C_1 \subset V$ if and only if $C_2 \subset V$.*

Proof. One can show by induction that there exists a point $x \in C_1 \cap \Gamma_m^{(n-1)}$. Since $\Omega_s^{(n-1)} \cap \{\omega_1 < X_n < \omega_2\} = \emptyset$, there exists $y \in C_2 \cap V$ such that $y \in \Gamma_m^{(n-1)}$. Thus, $C_2 \subset V$. □

For any α such that $\omega_1 < \alpha < \omega_2$ denote by C_α the cylindrical p-dimensional cell in $I_n^n(\alpha)$ having a multi-index $(i_1, \ldots, i_{n-1})$.

Lemma 5.3. *The union* $C = \bigcup_{\alpha\in(\omega_1,\omega_2)} C_\alpha$ *is a cylindrical* $(p{+}1)$*-dimensional cell in* I^n*.*

Proof. We prove by induction on n and p that C satisfies Definition 2.10. The base case of $n = 1$ is trivial. According to the construction and the inductive hypothesis, for each α there exists a cell C'_α in $I^n_n(\alpha) \cap \{X_1 = 0\}$ having the multi-index $(0, i_2, \ldots, i_{n-1})$. In particular, $\pi_2(C_\alpha) = C'_\alpha$. By the inductive hypothesis, the union $C' = \bigcup_{\alpha\in(\omega_1,\omega_2)} C'_\alpha$ is a cylindrical cell in $I^n_1(0)$. Also, by the inductive hypothesis, C_α is either a graph of a continuous bounded function $h_\alpha : C'_\alpha \to I^1$, or a sector between two such functions (see Definition 2.10). Let, for definiteness, C_α be the graph of some h_α. Then C is the graph of the continuous bounded function $h : C' \to I^1$ such that $h(x_2, \ldots, x_{n-1}, x_n) = h_{x_n}(x_2, \ldots, x_{n-1})$. □

Lemma 5.3 implies that either $C \subset V$ or $C \cap V = \emptyset$, i.e., $\mathcal{D}$ is compatible with V.

Lemma 5.4. *The decomposition* $\mathcal{D}$ *is a cylindrical cell decomposition.*

Proof. Straightforward. □

Remark 5.5. Let in the induction step n of the description of $\mathcal{D}$, the definition of the set $\Omega^{(n-1)}_s$ be modified by adding an arbitrary point $x \in cl(\Gamma^{(n-1)}_m)$. Thus,

$$\Omega^{(n-1)}_s \equiv S_n(\Gamma^{(n-1)}_0) \cup S_n(\Gamma^{(n-1)}_m) \cup \{x\}.$$

Clearly, this leads to another cylindrical cell decomposition of I^n compatible with V which is a refinement of $\mathcal{D}$.

6 Constructing a cell decomposition: stage I

The algorithm recursively constructs the decomposition $\mathcal{D}$ described in previous sections. The idea is to find the finite set $\Omega^{(n-1)}_0$ and thereby the cell decomposition $\mathcal{D}^{(n-1)}$ induced by $\mathcal{D}$ on $I^1 \subset \{X_1 = \cdots = X_{n-1} = 0\}$. For each cell C of $\mathcal{D}^{(n-1)}$ the algorithm finds the finite set $\Omega^{(n-2)}_0$ parameterized by the points of C, and thereby the cell decomposition $\mathcal{D}^{(n-2)}$ induced on $I^2 \subset \{X_1 = \cdots = X_{n-2} = 0\}$. On the last step of this recursion the parameterized set $\Omega^{(0)}_0$ and the cell decomposition $\mathcal{D} = \mathcal{D}^{(0)}$ of I^n will be found, that is compatible with V.

A straightforward representation of a finite (generally parametric) set $\Omega^{(i)}_0$, by means of formula $\Phi^{(i)}_0$ (see Section 4) would require quantifier alternation which we clearly want to avoid. In the following algorithm, at step i of the induction, for fixed values of coordinates $X_{i+1}, \ldots, X_n$, we represent $\Omega^{(i-1)}_0$ in the following three stages. Firstly, we approximate $\Omega^{(i-1)}_s$ by a finite (parametric) set of points defined by an *existential* formula. Then we define $\Omega^{(i-1)}_s$ by passing to limit with a help of Lemma 2.9. Finally, we define the

set $\Omega_0^{(i-1)}$ itself by an existential formula which involves the formula for the set $\Omega_s^{(i-1)}$.

Define recursively the sequence of integers $s_1, \ldots, s_n$ by setting $s_1 = 2$ and $s_{i+1} = 5s_i + 2i + 2$ for $1 \leq i \leq n-1$. Introduce new variables $Y_1, \ldots, Y_{s_n}, Z$.

Let $D_i \equiv (X_i \geq 0) \wedge (X_i \leq 1)$, so that $\{\bigwedge_{1 \leq i \leq k} D_i\} = I^k$. Denote $T^{(m)} = (T_1, \ldots, T_m)$, the m-tuple of variables T_i, $i \leq m$.

Let $X = X^{(n)}$. We now introduce formulae $G_0^{(i)}, G_m^{(i)}, G^{(i)}$ by induction on i. In cases $i = 1, 2$ we include comments explaining some non-trivial parts of the construction. Note that at a step i we treat $X_{i+1}, \ldots, X_n$ as parameters of formulae.

Step i = 1.

$f^{(0)}(X) \equiv (f(X))^2$
In case when $f \not\equiv 0$ the equation $f^{(0)}(X) = 0$ defines the set $\Omega_0^{(0)} \setminus \{0, 1\}$ and possibly some points outside $[0, 1]$.

$h_Z^{(0)}(X, Z) \equiv (f^{(0)}(X) - Z)^2$
The points satisfying $f^{(0)}(X) = 0$ are perturbed by Z.

$H_Z^{(0)} \equiv \big(h_Z^{(0)} = 0\big)$

$\Theta_0^{(0)} \equiv (cl\big(H_Z^{(0)} \wedge (Z > 0)\big) \wedge (Z = 0)) \vee (X_1 \cdot (X_1 - 1) = 0)$
This formula defines the limits of perturbed points as $Z \to +0$ with added $\{0, 1\}$, i.e, the set $\Omega_0^{(0)}$ and possibly some points outside $[0, 1]$. The only purpose of perturbation is to start the pattern which is meaningful on further induction steps.

$\Theta_m^{(0)} \equiv \big(X_1 = 1/2(Y_1 + Y_2) \wedge \Theta_0^{(0)}(Y_1, X_2, \ldots, X_n) \wedge \Theta_0^{(0)}(Y_2, X_2, \ldots, X_n) \wedge (f^{(0)}(X) = 0)\big)$
Defining the set $\Omega_m^{(0)}$ and possibly some points outside $[0, 1]$.

$G_0^{(1)}(Y^{(2)}, X) \equiv \Theta_0^{(0)} \wedge (Y_1 = Y_2 = 0) \wedge D_1$
Defining the set $\Omega_0^{(0)}$ and the (parametric) curve $\Gamma_0^{(1)}$ as projections along variables Y_1, Y_2.

$G_m^{(1)}(Y^{(2)}, X) \equiv \Theta_m^{(0)} \wedge D_1$
Defining the set $\Omega_m^{(0)}$ and the (parametric) curve $\Gamma_m^{(1)}$ as projections along variables Y_1, Y_2.

$G^{(1)}(Y^{(2)}, X) \equiv G_0^{(1)}(Y^{(2)}, X) \vee G_m^{(1)}(Y^{(2)}, X)$

Defining $\Gamma_0^{(1)} \cup \Gamma_m^{(1)}$.

Step i = 2.

$G_*^{(1)}(Y^{(2)}, X) \equiv \bigvee_{1 \leq l \leq M_1} \big((f_{l*}^{(1)}(Y^{(2)}, X) = 0) \wedge (g_{l*}^{(1)}(Y^{(2)}, X) > 0)\big)$ for $* \in \{0, m\}$.

Representing each $G^{(1)}_*(Y^{(2)}, X)$ as a Boolean combination of atomic equations and inequalities.

For each l, $1 \le l \le M_0$, define :
$h^{(1)}_{l,Z,*}(Y^{(4)}, X, Z) \equiv (f^{(1)}_{l*}(Y^{(2)}, X) - Z)^2 + \left(\frac{\partial f^{(1)}_{l*}}{\partial X_1}\right)^2 + \left(\frac{\partial f^{(1)}_{l*}}{\partial Y_1}\right)^2 + \left(\frac{\partial f^{(1)}_{l*}}{\partial Y_2}\right)^2 + Y_3^2 + Y_4^2$
For small values of Z the equation $f^{(1)}_{l*}(X) = Z$ defines a smooth hypersurface. Then $h^{(1)}_{l,Z,*}(Y^{(4)}, X, Z) = 0$ defines the set of all critical points of the coordinate function X_2 on this hypersurface. The purpose of introducing variables Y_3, Y_4 will be explained below.

$H^{(1)}_{Z*} \equiv \bigvee_{1 \le l \le M_1} \left((h^{(1)}_{l,Z,*} = 0) \wedge (g^{(1)}_{l*} > 0)\right)$
Collecting together the critical points on $f^{(1)}_{l*}(X) = Z$ for all l, $1 \le l \le M_1$ and selecting the ones which are relevant. Note that for small values of $Z > 0$ all points of local extrema of the coordinate function X_2 on $\{G^{(1)}_*(Y^{(2)}, X)\}$, except possibly the ones with $X_2(X_2 - 1) = 0$, are close to corresponding critical points.

$\Theta^{(1)}_{e*}(Y^{(4)}, X) \equiv (cl\big(H^{(1)}_{Z*} \wedge (Z > 0)\big) \wedge (Z = 0)) \vee (G^{(1)}_* \wedge (X_2(X_2 - 1) = 0))$
Passing to limit as $Z \to +0$ and adding $G^{(1)}_* \wedge (X_2(X_2 - 1) = 0)$ produces a finite (parameterized) set of points on $\{G^{(1)}_*\}$ which includes all points of local extrema of X_2 on $\{G^{(1)}_*\}$. The projection of $\{\Theta^{(1)}_{e*}(Y^{(4)}, X)\}$ along variables Y_1, Y_2, Y_3, Y_4 contains all points of local extrema of X_2 on $\Gamma^{(1)}_*$.

$\Theta^{(1)}_{\partial*}(Y^{(4)}, X) \equiv \partial(G^{(1)}_*(Y^{(2)}, X)) \wedge (Y_3 = Y_4 = 0)$
Defining a finite set of frontier points of $\{G^{(1)}_*(Y^{(2)}, X)\}$. The projection of $\{\Theta^{(1)}_{\partial*}(Y^{(4)}, X)\}$ along variables Y_1, Y_2, Y_3, Y_4 contains $B(\{\Gamma^{(1)}_*\})$.

$G^{(1)}_{1*} \equiv G^{(1)}_*(Y^{(2)}, X_1 - Z, X_2, \ldots, X_n)$
This defines a curve obtained from $\{G^{(1)}_*\}$ by shifting it along the coordinate axis X_1 by Z.

$Q^{(1)}_{Z*}(Y^{(4)}, X_1 - Z, X) \equiv G^{(1)}_{1*}(Y^{(2)}, X_1 - Z, X_2, \ldots, X_n) \wedge G^{(1)}_*(Y_3, Y_4, X)$
Intersecting the projection of $\{G^{(1)}_*\}$ with the projection of its shift produces a finite (parameterized) subset of $\Gamma^{(1)}_*$. Note that we need two additional variables Y_3, Y_4. Observe that for a small value $|Z|$ each ramification point of $\{\Gamma^{(1)}_*\}$ is close to the projection along Y_1, Y_2, Y_3, Y_4 of some of the points from $\{Q^{(1)}_{Z*}\}$.

$\Theta^{(1)}_{r*}(Y^{(4)}, X_1 - Z, X) \equiv cl\big(Q^{(1)}_{Z*} \wedge (Z > 0)\big) \wedge (Z = 0)$
Passing to limit as $Z \to +0$ produces a finite (parameterized) set of points on $\{G^{(1)}_*\}$ such that its projection along variables Y_1, Y_2, Y_3, Y_4, X_1 contains all X_2-coordinates of ramification points of $\Gamma^{(1)}_*$.

$\Theta^{(1)}_s(Y^{(4)}, X) \equiv \Theta^{(1)}_{e0} \vee \Theta^{(1)}_{\partial 0} \vee \Theta^{(1)}_{r0} \vee \Theta^{(1)}_{em} \vee \Theta^{(1)}_{\partial m} \vee \Theta^{(1)}_{rm}$

Defining a set whose projection along variables $Y_1, \ldots, Y_4, X_1$ is a finite set containing all X_2-coordinates of points from $S_2(\Gamma_0^{(1)}) \cup S_2(\Gamma_m^{(1)})$.

$\Theta_0^{(1)}(Y^{(7)}, X) \equiv G_0^{(1)}(Y^{(2)}, X) \wedge \Theta_s^{(1)}(Y_3, Y_4, \ldots, Y_7, X_2, \ldots, X_n)$
Defining a set whose projection along variables $Y_1, \ldots, Y_7$ contains $\Omega_0^{(1)}$. Note that in the expression $\Theta_s^{(1)}(Y_3, Y_4, \ldots, Y_7, X_2, \ldots, X_n)$ variables $Y_3, \ldots, Y_6$ stand for $Y^{(4)}$ in the definition of $\Theta_s^{(1)}$ while Y_7 stands for X_1. For any fixed values of parameters $X_3, \ldots, X_n$ the set $\Theta_s^{(1)}$ is finite and therefore the set $\{G_0^{(1)} \wedge \Theta_s^{(1)}\}$ reduces to an intersection of two finite unions of affine subspaces of complementary dimensions in 8-dimensional space. It follows that $\Theta_0^{(1)}(Y^{(7)}, X)$ is finite.

$$\Theta_m^{(1)}(Y^{(14)}, X) \equiv \big(X_2 = 1/2(Y_8 + Y_{14}) \wedge$$
$$G^{(1)}(Y^{(2)}, X) \wedge \Theta_s^{(1)}(Y_3, Y_4, \ldots, Y_8, X_3, \ldots, X_n) \wedge$$
$$\wedge \Theta_s^{(1)}(Y_9, \ldots, Y_{14}, X_3, \ldots, X_n)\big)$$

Defining a finite set of points whose projection along variables $Y_1, \ldots, Y_{14}$ contains $\Omega_m^{(1)}$.

$G_0^{(2)}(Y^{(14)}, X) \equiv \Theta_0^{(1)}(Y^{(7)}, X) \wedge (Y_8 = \cdots = Y_{14} = 0) \wedge D_2$
$G_m^{(2)}(Y^{(14)}, X) \equiv \Theta_m^{(1)}(Y^{(14)}, X) \wedge D_2$
$G^{(2)}(Y^{(14)}, X) \equiv G_0^{(2)}(Y^{(14)}, X) \vee G_m^{(2)}(Y^{(14)}, X)$

General step. Assume that on step i, $i \leq n-1$, the expression
$G^{(i)}(Y^{(s_i)}, X) \equiv G_0^{(i)} \vee G_m^{(i)}$
was defined. The interpretations of the following formulae are analogous to the ones provided in **step 2**.

Step (i + 1).

$G_*^{(i)}(Y^{(s_i)}, X) \equiv \bigvee_{1 \leq l \leq M_i} \big(\big(f_{l*}^{(i)}(Y^{(s_i)}, X) = 0\big) \wedge \big(g_{l*}^{(i)}(Y^{(s_i)}, X) > 0\big)\big)$, where $* \in \{0, m\}$.

For each l, $1 \leq l \leq M_i$, define :
$h_{l,Z,*}^{(i)}(Y^{(2s_i)}, X, Z) \equiv \big(f_{l*}^{(i)} - Z\big)^2 + \sum_{1 \leq j \leq i+1} \left(\frac{\partial f_{l*}^{(i)}}{\partial X_j}\right)^2 + \sum_{1 \leq j \leq s_i} \left(\frac{\partial f_{l*}^{(i)}}{\partial Y_j}\right)^2 +$
$\sum_{s_i+1 \leq j \leq 2s_i} \left(Y_j\right)^2$

$H_{Z*}^{(i)} \equiv \bigvee_{1 \leq l \leq M_i} \big((h_{l,Z,*}^{(i)} = 0) \wedge (g_{l*}^{(i)} > 0)\big)$

$\Theta_{e*}^{(i)} \equiv cl\big(H_{Z*}^{(i)} \wedge (Z > 0)\big) \wedge (Z = 0) \vee (G_*^{(i)} \wedge (X_{i+1}(X_{i+1} - 1)) = 0)$

$\Theta_{\partial*}^{(i)} \equiv \partial(G_*^{(i)}(Y^{(s_i)}, X)) \wedge (Y_{s_i+1} = \cdots = Y_{2s_i} = 0)$
$G_{1*}^{(i)} \equiv G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i-1}, X_i - Z, X_{i+1}, \ldots, X_n)$

$G_{2*}^{(i)} \equiv G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i-2}, X_{i-1} - Z, X_i, \ldots, X_n)$

$\cdots\cdots$

$G^{(i)}_{j*} \equiv G^{(i)}_*(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i-j}, X_{i+1-j} - Z, X_{i+2-j}, \ldots, X_n)$

$\cdots\cdots$

$G^{(i)}_{i*} \equiv G^{(i)}_*(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1 - Z, X_2, \ldots, X_n)$

$Q^{(i)}_{1,Z,*}(Y^{(2s_i)}, X_i - Z, X) \equiv G^{(i)}_{1*} \wedge G^{(i)}_*$

$Q^{(i)}_{2,Z,*}(Y^{(2s_i)}, X_{i-1} - Z, X) \equiv G^{(i)}_{2*} \wedge G^{(i)}_*$

$\cdots\cdots$

$Q^{(i)}_{j,Z,*}(Y^{(2s_i)}, X_{i+1-j} - Z, X) \equiv G^{(i)}_{j*} \wedge G^{(i)}_*$

$\cdots\cdots$

$Q^{(i)}_{i,Z,*}(Y^{(2s_i)}, X_1 - Z, X) \equiv G^{(i)}_{i*} \wedge G^{(i)}_*$

$Q^{(i)}_{Z*}(Y^{(2s_i)}, X_1 - Z, \ldots, X_i - Z, X) \equiv \bigvee_{1\le j\le i} \big(Q^{(i)}_{j,Z,*}\big)$

$\Theta^{(i)}_{r*} \equiv cl\big(Q^{(i)}_{Z*} \wedge (Z > 0)\big) \wedge (Z = 0)$

$\Theta^{(i)}_s(Y^{(2s_i)}, X) \equiv \Theta^{(i)}_{e0} \vee \Theta^{(i)}_{\partial 0} \vee \Theta^{(i)}_{r0} \vee \Theta^{(i)}_{em} \vee \Theta^{(i)}_{\partial m} \vee \Theta^{(i)}_{rm}$

$\Theta^{(i)}_0(Y^{(3s_i+i)}, X) \equiv G^{(i)}_0(Y^{(s_i)}, X) \wedge$
$\quad \wedge \Theta^{(i)}_s(Y_{s_i+1}, \ldots, Y_{3s_i}, Y_{3s_i+1}, \ldots, Y_{3s_i+i}, X_{i+1}, \ldots, X_n)$

$\Theta^{(i)}_m(Y^{(s_{i+1})}, X) \equiv (X_{i+1} = 1/2(Y_{3s_i+i+1} + Y_{5s_i+2i+2}) \wedge G^{(i)}(Y^{(s_i)}, X) \wedge$
$\quad \wedge \Theta^{(i)}_s(Y_{s_i+1}, \ldots, Y_{3s_i+i+1}, X_{i+2}, \ldots, X_n) \wedge$
$\quad \wedge \Theta^{(i)}_s(Y_{3s_i+i+2}, \ldots, Y_{5s_i+2i+2}, X_{i+2}, \ldots, X_n)$

$G^{(i+1)}_0(Y^{(s_{i+1})}, X) \equiv \Theta^{(i)}_0(Y^{(3s_i+i)}, X) \wedge (Y_{3s_i+i+1} = \cdots = Y_{s_{i+1}} = 0) \wedge D_{i+1}$

$G^{(i+1)}_m(Y^{(s_{i+1})}, X) \equiv \Theta^{(i)}_m(Y^{(s_{i+1})}, X) \wedge D_{i+1}$

$G^{(i+1)}\big(Y^{(s_{i+1})}, X\big) \equiv G^{(i+1)}_0 \vee G^{(i+1)}_m$

End of the general step.

For each i, $1 \le i \le n$ let $\rho_i : \mathbb{R}^{s_i+i} \to \mathbb{R}^i$ be the projection map along $Y^{(s_i)}$ onto the subspace with coordinates $X_1, \ldots, X_i$. Consider a vector $(\omega_{i+2}, \ldots, \omega_n)$ such that $0 \le \omega_j \le 1$ for all $j = i+2, \ldots, n$. For any $* \in \{0, m\}$ let $\Omega^{(i)}_*(\omega_{i+2}, \ldots, \omega_n)$ denote the set $\Omega^{(i)}_*$ for $V \cap \{X_{i+2} = \omega_{i+2}, \ldots, X_n = \omega_n\}$ in the cube I^{n-i+1} identified with $I^n \cap \{X_{i+2} = \omega_{i+2}, \ldots, X_n = \omega_n\}$.

Lemma 6.1. *For any $* \in \{0, m\}$ the projection $\rho_{i+1}(\{\Theta^{(i)}_*(Y^{(3s_i+i)}, X)\} \cap \{X_{i+2} = \omega_{i+2}, \ldots, X_n = \omega_n\})$ is a finite set of points containing $\Omega^{(i)}_*(\omega_{i+2}, \ldots, \omega_n)$.*

Proof. The proof is a straightforward routine using induction on i. For $i = 2$ it is actually contained in the comments to formulae defining $\Theta^{(1)}_0$, $\Theta^{(1)}_m$ (see **step i = 2**). □

Lemma 6.2. *For each $k \in \{1, \ldots, n\}$ the format of the sets defined by $G_*^{(k)}$ (for $* \in \{0, m\}$) is $(N_k, \alpha, \beta_k, r_k, m_k)$, where*

$$N_k = (\alpha + \beta N)^{(r+n)^{O(k)} 2^{O(k^2)}}, \beta_k = (\alpha + \beta N)^{(r+n)^{O(k)} 2^{O(k^2)}},$$

$$r_k = r5^{k-1}, m_k = O(n5^k).$$

Proof. Recall that the semi-Pfaffian set $V = (\{f = 0\} \cap I^n) \cup I_1^n$ has format $(O(n2^n), \alpha, 2\beta N, r, n)$. Introduce notations $N_V = O(n2^n)$ and $D_V = O(\alpha + \beta N)$.

At step k, $1 \leq k \leq n$, let $m_k = s_k + n$ be the total number of variables and r_k the size of the Pfaffian chain for the functions in $G_*^{(k)}$. Recall that $s_1 = 2$ and $s_k = 5s_{k-1} + 2k$. Then $s_k = O(k5^k)$ and $m_k = O(n5^k)$.

For $k = 1$ the order $r_1 = r$ since the operation of taking closure (see Lemma 2.9) leaves the size of a Pfaffian chain unchanged. Suppose that $F(Y^{(s_{k-1})}, X) = \big(f_1(Y^{(s_{k-1})}, X), \ldots, f_{r_{k-1}}(Y^{(s_{k-1})}, X)\big)$ is the Pfaffian chain of the set defined by $G_*^{(k-1)}$. Notice that the order of the Pfaffian chain of the set defined by $Q_{j,Z,*}^{(k-1)}$, $1 \leq j \leq k-1$ is $2r_{k-1}$ since we need to add in this chain the same functions as before but with variables $Y^{(s_{k-1})}, X_{k+1-j}$ replaced by $Y_{s_{k-1}+1}, \ldots, Y_{2s_{k-1}}, X_{k+1-j} - Z$ respectively. Thus, the size of the Pfaffian chain of $Q_{Z*}^{(k-1)}$ is equal to kr_{k-1}. According to Lemma 2.9, there is a Boolean formula, say $Q_*^{(k-1)}$, with atomic Pfaffian functions in variables $Y^{(2s_{k-1})}, X_1 - Z, \ldots, X_{k-1} - Z, X, Z$, having the same common Pfaffian chain as $Q_{Z*}^{(k-1)}$ such that $\{Q_*^{(k-1)}\} = cl\big\{Q_{Z*}^{(k-1)} \wedge (Z > 0)\big\}$. The formula $\Theta_{r*}^{(k-1)} \equiv Q_*^{(k-1)} \wedge (Z = 0)$ is equivalent to an expression involving Pfaffian functions only in variables $Y^{(2s_{k-1})}, X$. Substituting the value 0 for the variable Z in every function present in the Pfaffian chain of $Q_*^{(k-1)}$ we see that the Pfaffian chain of $\Theta_{r*}^{(k-1)}$ is $F(Y^{(s_{k-1})}, X), F(Y_{s_{k-1}+1}, \ldots, Y_{2s_{k-1}}, X)$. Similarly, formulae $\Theta_{e*}^{(k-1)}$ and $\Theta_{\partial*}^{(k-1)}$ are equivalent to expressions involving Pfaffian functions only in variables $Y^{(2s_{k-1})}, X$, having common Pfaffian chain $F(Y^{(s_{k-1})}, X)$. Thus, the Pfaffian chain of $\Theta_s^{(k-1)}$ is again $F(Y^{(s_{k-1})}, X), F(Y_{s_{k-1}+1}, \ldots, Y_{2s_{k-1}}, X)$. It follows that the Pfaffian chain of $\Theta_0^{(k-1)}$ is

$$F(Y^{(s_{k-1})}, X), F(Y_{s_{k-1}+1}, \ldots, Y_{2s_{k-1}}, Y_{3s_{k-1}+1}, \ldots, Y_{3s_{k-1}+k-1}, X_k, \ldots, X_n),$$

$$F(Y_{2s_{k-1}+1}, \ldots, Y_{3s_{k-1}}, Y_{3s_{k-1}+1}, \ldots, Y_{3s_{k-1}+k-1}, X_k, \ldots, X_n)$$

and the common Pfaffian chain of $\Theta_m^{(k-1)}$ and $G_*^{(k)}$ is

$$F(Y^{(s_{k-1})}, X), F(Y_{s_{k-1}+1}, \ldots, Y_{2s_{k-1}}, Y_{3s_{k-1}+1}, \ldots, Y_{3s_{k-1}+k}, X_{k+1}, \ldots, X_n),$$

$$F(Y_{2s_{k-1}+1}, \ldots, Y_{3s_{k-1}}, Y_{3s_{k-1}+1}, \ldots, Y_{3s_{k-1}+k}, X_{k+1}, \ldots, X_n),$$

$$F(Y_{3s_{k-1}+k+1}, \ldots, Y_{4s_{k-1}+k}, Y_{5s_{k-1}+k+1}, \ldots, Y_{5s_{k-1}+2k}, X_{k+1}, \ldots, X_n),$$

$$F(Y_{4s_{k-1}+k+1},\dots,Y_{5s_{k-1}+k},Y_{5s_{k-1}+k+1},\dots,Y_{5s_{k-1}+2k},X_{k+1},\dots,X_n).$$

We conclude that the order of the set defined by $G_*^{(k)}$ is $r_k = 5r_{k-1} = r5^{k-1}$.

Let

$$p_k = \prod_{1\le j\le k-1} m_j = n^{k-1}2^{O(k^2)}, \quad q_k = \prod_{1\le j\le k-1}(m_j+r_j) = (r+n)^{k-1}2^{O(k^2)}.$$

For $k=1$, applying the bounds from Lemma 2.9 we get

$$N_1 = (2^{r^2}N_V)^{O(m_1(m_1+r_1))}(m_1D_V)^{O(m_1(m_1+r_1)^2)}$$

and

$$\beta_1 = 2^{m_1r_1^2}(m_1D_V)^{O(m_1(m_1+r_1))}$$

Note that at each other step we perform two iterations of the closure operation. It can be seen that

$$\beta_k = (2^r p_k D_V)^{O(p_k^2q_k^2)} = D_V^{(r+n)^{O(k)}2^{O(k^2)}}$$

and

$$N_k = (2^{r^2}p_kN_V\beta_k^{2k})^{O(p_k^2q_K^3)} = D_V^{(r+n)^{O(k)}2^{O(k^2)}}.$$

□

The algorithm writes out formulae $G_*^{(k)}$, $G^{(k)}$ for all $1\le k\le n$ using the described recursive formulae. The complexity of this stage of the algorithm does not exceed $(\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}}$.

7 Constructing a cell decomposition: stage II

The second stage of the algorithm consists of the following recursive procedure. For $0\le i\le n-1$ and for fixed values of coordinates $X_{i+2},\dots,X_n$ let $\Omega_L^{(i)} = \Omega_0^{(i)}\cap L_i^n(0)\subset\Omega_0^{(i)}$. This is the projection of $\Omega_0^{(i)}$ on $\{X_1=\dots=X_i=0\}$. According to Lemma 6.1, $\Omega_L^{(i)}\subset\rho_{i+1}(\{G_0^{(i+1)}(Y^{(s_i)},X)\wedge(X_1=\dots=X_i=0)\})$. Introduce the notation

$$\Psi_L^{(i)}(X)\equiv(\exists Y^{(s_i)})\big(G_0^{(i)}(Y^{(s_i)},X)\wedge(X_1=\dots=X_i=0)\big).$$

Thus, $\Omega_L^{(i)}\subset\{\Psi_L^{(i)}(X)\}$.

For each $i\in\{0,\dots,n\}$ we define as follows a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of $L_i^n(0)$ which is compatible with the projection of V onto the subset $L_i^n(0)$ of $\mathbb{R}^{n-i}$ equipped with coordinates $X_{i+1},\dots,X_n$.

For each $k\in\{0,\dots,n\}$ let $i=n-k$. We proceed by induction on k.

For $k=0$ set $\alpha=(0,\dots,0)$ and $C_\alpha^{(n)}=\{(0,\dots,0)\}$ to be the only cell of the cylindrical cell decomposition $\mathcal{D}^{(n)}$ of $L_n^n(0)$.

Suppose that on the step k a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of the cube $L_i^n(0)$ was defined. On the next step $k+1$ (with $i = n-k-1$) the input is the cylindrical cell decomposition $\mathcal{D}^{(i+1)}$ of the cube $L_{i+1}^n(0)$ obtained on the previous step. For each cell $C_\alpha^{(i+1)} \in \mathcal{D}^{(i+1)}$ denote by $Z(C_\alpha^{(i+1)})$ the set $C_\alpha^{(i+1)} \times [0,1]$, which is the bounded cylinder over $C_\alpha^{(i+1)}$ and along X_{i+1}, contained in the cube $L_i^n(0)$. The algorithm constructs a cell decomposition of $Z(C_\alpha^{(i+1)})$ in the following way. Observe that for any point $z = (z_{i+2}, \ldots, z_n) \in C_\alpha^{(i+1)}$, the cardinality of $\{\Psi_L^{(i)}(X_1, \ldots, X_{i+1}, z)\}$ is finite and constant over $C_\alpha^{(i+1)}$. Corollary 2.7 implies that this number does not exceed

$$M_i = (\alpha + \beta N)^{(r+n)^{O(i)} 2^{O(i^2)}}.$$

Introduce new variables $x_{j,i+1}^{(i)}$, $1 \le j \le M_i$, and denote $x_j^{(i)} = (0, \ldots, 0, x_{j,i+1}^{(i)}, z)$.

The algorithm finds the exact number K_α of distinct points in

$$\{\Psi_L^{(i)}(X_1, \ldots, X_{i+1}, z)\}$$

by testing with the oracle for each m, $1 \le m \le M_i$, whether the statement

$$(\exists z)(\exists x_{1,i+1}^{(i)}) \cdots (\exists x_{m,i+1}^{(i)}) \Big[\Big(\bigwedge_{1 \le j \le m} \Psi_L^{(i)}(x_j^{(i)}) \Big) \wedge$$

$$\Big(\bigwedge_{1 \le r \le m-1} \bigwedge_{1 \le j \le r} (x_{r+1,i+1}^{(i)} \ne x_{j,i+1}^{(i)}) \Big) \Big]$$

is true. The number K_α is the maximal m for which the statement holds.

We now describe all cells in $Z(C_\alpha^{(i+1)}) \subset L_i^n(0)$ by the following formulae.

- Sections over $C_\alpha^{(i+1)}$: for $1 \le j \le K_\alpha$

$$C_{\alpha'}^{(i)} = \Big\{ (0, \ldots, 0, z_{i+1}, \ldots, z_n) \in Z(C_\alpha^{(i+1)}) \ : \ (\exists x_1^{(i)}) \cdots (\exists x_{K_\alpha}^{(i)}) \Big(\bigwedge_{1 \le j \le K_\alpha} \Psi_L^{(i)}(x_j^{(i)}) \wedge$$

$$\wedge x_{1,i+1}^{(i)} < \cdots < x_{j-1,i+1}^{(i)} < x_{j,i+1}^{(i)} = z_{i+1} < x_{j+1,i+1}^{(i)} < \cdots < x_{K_\alpha,i+1}^{(i)} \Big) \Big\},$$

  where the index $\alpha' = (0, \ldots, 0, 2j-2, \alpha_{i+2}, \ldots, \alpha_n)$.

- Sectors over $C_\alpha^{(i+1)}$: for $1 \le j \le K_\alpha - 1$

$$C_{\alpha'}^{(i)} = \Big\{ (0, \ldots, 0, z_{i+1}, \ldots, z_n) \in Z(C_\alpha^{(i+1)}) \ : \ (\exists x_1^{(i)}) \cdots (\exists x_{K_\alpha}^{(i)}) \Big(\bigwedge_{1 \le j \le K_\alpha} \Psi_L^{(i)}(x_j^{(i)}) \wedge$$

$$\wedge x^{(i)}_{1,i+1} < \cdots < x^{(i)}_{j-1,i+1} < x^{(i)}_{j,i+1} < z_{i+1} < x^{(i)}_{j+1,i+1} < \cdots < x^{(i)}_{K_\alpha,i+1}) \Big\},$$

where the index $\alpha' = (0, \ldots, 0, 2j-1, \alpha_{i+2}, \ldots, \alpha_n)$.

Combining the cell decompositions of $Z(C^{(i+1)}_\alpha)$ for all cells $C^{(i+1)}_\alpha$ in $\mathcal{D}^{(i+1)}$, we obtain a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of the cube $L^n_i(0)$. This concludes the description of the algorithm.

We conclude by estimating the complexity of the algorithm. On each recursion step $k = 0, \ldots, n$ of the second stage of the algorithm, the degrees of Pfaffian functions remain the same as in $G^{(n-k)}_0$, while the numbers of variables, orders and atomic formulae are multiplied by at most $M_{n-k} + 1$ each. The number of cells is increased on a recursion step according to the formula $|\mathcal{D}^{(n-k)}| \leq O(M_{n-k})|\mathcal{D}^{(n-k+1)}|$ and $|\mathcal{D}^{(n)}| = 1$, so $|\mathcal{D}^{(0)}| \leq O(M_n)^n$.

It follows that the described two-stage algorithm produces a cylindrical cell decomposition $\mathcal{D}^{(0)}$ of I^n, compatible with the semi-Pfaffian set $S \cap I^n$, consisting of $\Lambda = (\alpha + \beta N)^{r^{O(n)} 2^{O(n^2)}}$ cells. Each cell is a sub-Pfaffian set having a format $(\Lambda, \alpha, \Lambda, \Lambda, \Lambda)$. The complexity of the algorithm does not exceed Λ.

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About Authors

Savvas Pericleous (*mapsp@cs.bath.ac.uk*) and
Nicolai Vorobjov (*nnv@cs.bath.ac.uk*)
are at the Department of Computer Science, University of Bath, Bath, BA2 7AY, UK.

Acknowledgments

This work was supported in part by EPSRC grant GR/L77928. We are grateful to Andrei Gabrielov and Dan Richardson for useful discussions.