BETTI NUMBERS OF SEMIALGEBRAIC
AND SUB-PFAFFIAN SETS

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Abstract

Let $X$ be a subset in $[-1, 1]^n \subset \mathbb{R}^n$ defined by a formula

$$X = \{x_0 | Q_1 x_1 Q_2 x_2 \ldots Q_\nu x_\nu ((x_0, x_1, \ldots, x_\nu) \in X_\nu)\},$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $x_i \in \mathbb{R}^n$, and $X_\nu$ be either an open or a closed set in $[-1, 1]^n + \ldots + n_\nu$ being a difference between a finite CW-complex and its subcomplex. We express an upper bound on each Betti number of $X$ via a sum of Betti numbers of some sets defined by quantifier-free formulae involving $X_\nu$.

In important particular cases of semialgebraic and semi-Pfaffian sets defined by quantifier-free formulae with polynomials and Pfaffian functions respectively, upper bounds on Betti numbers of $X_\nu$ are well known. Our results allow to extend the bounds to sets defined with quantifiers, in particular to sub-Pfaffian sets.

Introduction

Well-known results of Petrovskii, Oleinik [16], [15], Milnor [13], and Thom [19] provide an upper bound for the sum of Betti numbers of a semialgebraic set defined by a Boolean combination of polynomial equations and inequalities. A refinement of these results can be found in [1]. For semi-Pfaffian sets the analogous bounds were obtained by Khovanskii [11] (see also [23]). In this paper we describe a reduction of estimating Betti numbers of sets defined by formulae with quantifiers to a similar problem for sets defined by a quantifier-free formula.

More precisely, let $X$ be a subset in $[-1, 1]^n \subset \mathbb{R}^n$ defined by a formula

$$X = \{x_0 | Q_1 x_1 Q_2 x_2 \ldots Q_\nu x_\nu ((x_0, x_1, \ldots, x_\nu) \in X_\nu)\},$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $x_i \in \mathbb{R}^n$, and $X_\nu$ be either an open or a closed set in $[-1, 1]^n + \ldots + n_\nu$ being a difference between a finite CW-complex and one of its subcomplexes. For instance, if $\nu = 1$ and $Q_1 = \exists$, then $X$ is the projection of $X_\nu$.

We express an upper bound on each Betti number of $X$ via a sum of Betti numbers of some sets defined by quantifier-free formulae involving $X_\nu$. In conjunction with Petrovskii-Oleinik-Thom-Milnor’s result this implies a new upper bound for semialgebraic sets defined by formulae with quantifiers, which is significantly better than a bound following from the cylindrical cell decomposition approach. In conjunction with Khovanskii’s result our method produces an analogous upper bound for restricted sub-Pfaffian sets defined by formulae with quantifiers. Apparently in this case no general upper bounds were previously known.

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Throughout the paper each topological space is assumed to be a difference between a finite CW-complex and one of its subcomplexes.

**Example 1.** The closure $X$ of the interior of a compact set $Y \subset [-1,1]^n$ is homotopy equivalent to

$$X_{\varepsilon,\delta} = \{ x \mid \exists y (\|x - y\| \leq \delta) \forall z (\|y - z\| < \varepsilon) (z \in Y) \}$$

for small enough $\delta, \varepsilon > 0$ such that $\delta \gg \varepsilon$. Representing $X_{\varepsilon,\delta}$ in the form (0.1), we conclude that $X$ is homotopy equivalent to $X_{\varepsilon,\delta} = \{ x \mid \exists y \forall z \} \circ \{ x \mid \exists y \forall z \}$, where $X_2 = \{ (x,y,z) \mid (\|x - y\| \leq \delta \land (\|y - z\| \geq \varepsilon \lor z \in Y) \}$ is a closed set in $[-1,1]^3n$. Our results allow to bound from above Betti numbers of $X$ in terms of Betti numbers of $X_2$.

**1. A spectral sequence associated with a surjective map**

**Definition 1.** A continuous map $f : X \to Y$ is locally split if for any $y \in Y$ there is an open neighbourhood $U$ of $y$ and a section $s : U \to X$ of $f$ (i.e., $s$ is continuous and $fs = Id$). In particular, a projection of an open set in $\mathbb{R}^n$ on a subspace of $\mathbb{R}^n$ is always locally split.

**Definition 2.** For two maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the fibered product of $X_1$ and $X_2$ is defined as

$$X_1 \times_Y X_2 := \{ (x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2) \}.$$  

**Theorem 1.** Let $f : X \to Y$ be a surjective cellular map. Assume that $f$ is either closed or locally split. Then for any Abelian group $G$, there exists a spectral sequence $E_{p,q}$ converging to $H_*(Y, G)$ with

$$E_{p,q}^1 = H_q(W_p, G)$$

where

$$W_p = X \times_Y \ldots \times_Y X$$

$p+1$ times

In particular,

$$\dim H_k(Y, G) \leq \sum_{p+q=k} \dim H_q(W_p, G),$$

for all $k$.

For a locally split map $f$, this theorem can be derived from [5], Corollary 1.3. We present below a proof for a closed map $f$.

**Remark 1.** In the sequel we use Theorem 1 only for projections of either closed or open sets in $[-1,1]^n$. If $f$ is a projection of an open set, then (1.3) easily follows from the analogous result for closed maps which will be proved below, without references to [5]. Indeed, for an open set $Z$ define its shrinking $S(Z)$ as the closed set $Z \setminus N(\partial Z)$ where $N$ denotes an open neighbourhood. For a small enough $N(\partial Z)$, the set $Z$ is homotopy equivalent to $S(Z)$ (recall that $Z$ is a difference between a finite
Let $X$ be open and $S(X)$ be its shrinking with a sufficiently small $N(\partial X)$. It induces shrinkings $S(Y) = f(S(X))$ and $S(W_p) = S(X) \times S(Y) \cdots \times S(Y) S(X)$ which are homotopy equivalent to $Y$ and $W_p$ respectively. The statement for open sets $X$ and $Y$ follows from the statement for closed sets applied to $f : S(X) \to S(Y)$.

**Definition 3.** For a sequence $(P_0, \ldots, P_p)$ of topological spaces, their *join* $P_0 \ast \cdots \ast P_p$ can be defined as follows. Let $\Delta^p = \{s_0 \geq 0, \ldots, s_p \geq 0, s_0 + \cdots + s_p = 1\}$ be the standard $p$-simplex. Then $P_0 \ast \cdots \ast P_p$ is the quotient space of $P_0 \times \cdots \times P_p \times \Delta^p$ over the following relation:

$$(x_0, \ldots, x_p, s) \sim (x'_0, \ldots, x'_p, s) \text{ if } s = (s_0, \ldots, s_p) \text{ and } x_i = x'_i \text{ whenever } s_i \neq 0.$$ (1.4)

Given a continuous surjective map $f_i : P_i \to Y$ for each $i = 0, \ldots, p$, the fibered join $P_0 \ast_Y \cdots \ast_Y P_p$ is defined as the quotient space of $P_0 \times_Y \cdots \times_Y P_p \times \Delta^p$ over the relation (1.4).

**Definition 4.** For a space $Z$, $1$-st suspension of $Z$ is defined as the suspension (see [12]) of $(Z \cup \{\text{point}\})$. For an integer $p > 0$, the $p$-th iteration of this operation will be called $p$-th suspension of $Z$.

**Lemma 1.** Let $f_i : P_i \to Y$, $i = 0, \ldots, p$, be continuous surjective maps and $P = P_0 \ast_Y \cdots \ast_Y P_p$ their fibered join. There is a natural map $F : P \to Y$ induced by the maps $f_0, \ldots, f_p$. For a point $y \in Y$ the fiber $F^{-1}y$ coincides with the join $f_0^{-1}y \ast \cdots \ast f_p^{-1}y$ of the fibers of $f_i$.

There is a natural map $\pi : P \to \Delta^p$. The fiber of $\pi$ over an interior point of $\Delta^p$ is $P_1 \times_Y \cdots \times_Y P_p$. For each $i = 0, \ldots, p$, there is a natural embedding

$$\phi_i : P(i) = P_0 \ast_Y \cdots \ast_Y P_{i-1} \ast_Y P_{i+1} \ast_Y \cdots \ast_Y P_p \to P.$$ (1.5)

Its image coincides with $\pi^{-1}(\{s_i = 0\})$, and the space $P/\left(\bigcup_i \phi_i(P(i))\right)$ is homotopy equivalent to the $p$-th suspension of $P_0 \times_Y \cdots \times_Y P_p$.

**Proof.** Directly follows from Definitions 3, 4.

**Definition 5.** Let $f : X \to Y$ be a surjective continuous map. Its *join space* $J^f(X)$ is the quotient space of the disjoint union of spaces

$$J^f_p(X) = X \ast_Y \cdots \ast_Y X, \quad p = 0, 1, \ldots,$$ (1.6)

identifying $J^f_{p-1}(X)$ with each of its images $\phi_i(J^f_{p-1}(X))$ in $J^f_p(X)$ for $i = 0, \ldots, p$, where $\phi_i$ is defined in (1.5). When $Y$ is a point, we write $J_p(X)$ instead of $J^f_p(X)$ and $J(X)$ instead of $J^f(X)$.

**Lemma 2.** Let $\phi : J_p(X) \to J(X)$ be the natural map induced by the maps $\phi_i$. Then $\phi(J_{p-1}(X))$ is contractible in $\phi(J_p(X))$.

**Proof.** Let $x$ be a point in $X$. For $t \in [0, 1]$, the maps
g(t, x_1, \ldots, x_p, s) \mapsto (x, x_1, \ldots, x_p, 1 - t + ts_0, ts_1, \ldots, ts_p)

are homotopy equivalent to $x \ast_x \cdots \ast_x x$.
define a contraction of $\phi_0(J_{p-1}(X))$ to the point $x \in X$ where $X$ is identified with its embedding in $J_p(X)$ as $\pi^{-1}(1,0,\ldots,0)$. It is easy to see that the maps $g_t$ are compatible with the equivalence relations in Definition 5 and define a contraction of $\phi(J_{p-1}(X))$ to a point in $\phi(J_p(X))$.

**Lemma 3.** The join space $J(X)$ is homologically trivial.

**Proof.** Any cycle in $J(X)$ belongs to $\phi(J_p(X))$ for some $p$, while according to Lemma 2 $\phi(J_p(X))$ is contractible in $J(X)$. Hence the cycle is homologous to 0. \qed

**Proof of Theorem 1.** Let $f$ be closed. Let $F : J^f(X) \rightarrow Y$ be the natural map induced by $f$. Then $F$ is also closed. Its fiber $F^{-1}y$ over a point $y \in Y$ coincides with the join space $J(f^{-1}y)$ which is homologically trivial according to Lemma 3. It follows that $\tilde{H}^*(J(f^{-1}y)) \cong 0$, where $\tilde{H}^*$ is the Alexander cohomology ([18], p. 308), since $\tilde{H}^*(Z) \cong H^*(Z)$ for any locally contractible space $Z$ ([18], p. 340), in particular for a difference between CW-complex and a subcomplex.

Vietoris-Begle theorem ([18], p. 344) applied to $F : J^f(X) \rightarrow Y$, implies

$$\tilde{H}^*(J^f(X), G) \cong H^*(Y, G)$$

and therefore

$$H_*(J^f(X), G) \cong H_*(Y, G).$$

By Lemma 1, the space $J^f_p(X)/ \left( \bigcup_{q<p} J^f_q(X) \right)$ is homotopy equivalent to the $p$-th suspension of $W_p$. Theorem 1 follows now from the spectral sequence associated with filtration of $J^f(X)$ by the spaces $J^f_p(X)$.

**Remark 2.** For a map $f$ with 0-dimensional fibers, a similar spectral sequence, “image computing spectral sequence” was applied to problems in theory of singularities and topology by Vassiliev [20], Goryunov-Mond [8], Goryunov [7], Houston [10], and others. For proper maps an analogous “cohomological descent spectral sequence” appears in [4].

**Remark 3.** A continuous map $f : X \rightarrow Y$ is called compact-covering if any compact set in $Y$ is an image of a compact set in $X$. This condition includes both the closed and the locally split cases and may be more convenient for applications. For a surjective cellular compact-covering $f : X \rightarrow Y$ Theorem 1 is also true. A proof will appear elsewhere.

## 2. Alexander’s duality and Mayer-Vietoris inequality

Let

$$I^n_i := \bigcap_{1 \leq j \leq n} \left\{ -i \leq x_j \leq i \right\} \subset \mathbb{R}^n.$$ 

Define the “thick boundary” $\partial I^n_i := I^n_{i+1} \setminus I^n_i$. The following lemma is a version of Alexander’s duality theorem.
Lemma 4. (Alexander’s duality) If $X \subset I^n_1$ is an open set in $I^n_1$, then for any $q \in \mathbb{Z}$, $q \leq n - 1$,

$$H_q(I^n_1 \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \partial I^n_1, \mathbb{R}).$$ \hspace{1cm} (2.1)

If $X \subset I^n_1$ is a closed set in $I^n_1$, then for any $q \in \mathbb{Z}$, $q \leq n - 1$,

$$H_q(I^n_1 \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \text{closure}(\partial I^n_1), \mathbb{R}).$$ \hspace{1cm} (2.2)

Proof. For definiteness let $X$ be closed. Compactifying $\mathbb{R}^n$ at infinity as $\mathbb{R}^n \cup \infty \simeq S^n$, we have, by Alexander’s duality [12],

$$\tilde{H}_q(S^n \setminus (X \cup \text{closure}(\partial I^n_1)), \mathbb{R}) \cong \tilde{H}_{n-q-1}((X \cup \text{closure}(\partial I^n_1)), \mathbb{R}).$$

The first group is isomorphic to $H_q(I^n_1 \setminus X, \mathbb{R})$ when $q > 0$, and to $H_0(I^n_1 \setminus X, \mathbb{R}) + \mathbb{R} \cong H_0(I^n_1 \setminus X, \mathbb{R})$ when $q = 0$. Combining these two cases, we obtain (2.2). \hfill \Box

Lemma 5. (Mayer-Vietoris inequality) Let $X_1, \ldots, X_m \subset I^n_1$ be all open or all closed in $I^n_1$. Then

$$b_i \left( \bigcup_{1 \leq j \leq n} X_j \right) \leq \sum_{J \subset \{1, \ldots, n\}} b_{i-|J|+1} \left( \bigcap_{j \in J} X_j \right)$$

and

$$b_i \left( \bigcap_{1 \leq j \leq n} X_j \right) \leq \sum_{J \subset \{1, \ldots, n\}} b_{i+|J|-1} \left( \bigcup_{j \in J} X_j \right),$$

where $b_i$ is the $i$th Betti number.

Proof. A well-known corollary to Mayer-Vietoris sequence. \hfill \Box

3. Thom-Milnor’s and Khovanskii’s bounds

Necessary definitions regarding semi-Pfaffian and sub-Pfaffian sets can be found in [11], [6]. In this paper we consider only restricted sub-Pfaffian sets.

To apply our results to semialgebraic sets and to restricted sub-Pfaffian sets, defined by formulae with quantifiers, we need the following known upper bounds on Betti numbers for sets defined by quantifier-free formulae.

Let $X = \{ \varphi \} \subset I^n_1$ be a semialgebraic set, where $\varphi$ is a Boolean combination with no negations of $s$ atomic formulae of the kind $f > 0$, $f$ being polynomials in $n$ variables with coefficients in $\mathbb{R}$, $\deg(f) < d$. We will refer to the sequence $(n, s, d)$ as to format of $\varphi$. It follows from [19], [13], [1] that the sum of Betti numbers of $X$ is

$$b(X) \leq O(sd)^n.$$ \hspace{1cm} (3.1)

If $X = \{ \varphi \}$ is a compact semialgebraic set, where $\varphi$ is a Boolean combination with no negations of $s$ atomic formulae of the kind either $f \geq 0$ or $f > 0$, $f$ being polynomials in $n$ variables, $\deg(f) < d$, then a combination of results from [19], [13], [1] and [14], [22] implies that that the sum of Betti numbers of $X$ also satisfies (3.1).

Now let $X = \{ \varphi \} \subset I^n_1$ be a semi-Pfaffian set, where $\varphi$ is a Boolean combination with no negations of $s$ atomic formulae of the kind $f > 0$, $f$ being Pfaffian functions in an open domain $G \supset I^n_1$ of order $\rho$, degree $(\alpha, \beta)$, having a common Pfaffian chain
with coefficients in \( \mathbb{R} \). The sequence \((n, s, \alpha, \beta, \rho)\) is called format of \( \varphi \). It follows from \cite{11, 23} that the sum of Betti numbers of \( X \) is

\[
b(X) \leq s^n2^{\rho(\rho-1)/2}O(n^{\beta} + \min\{n, \rho\}n^{\alpha + \rho}). \tag{3.2}
\]

Let \( X \subset I_1^{\infty} \) be a semialgebraic set defined by a formula

\[
Q_1x_1Q_2x_2 \ldots Q_nx_nF(x_0, x_1, \ldots, x_n), \tag{3.3}
\]

where \( Q_i \in \{\exists, \forall\} \), \( Q_i \neq Q_{i+1}, x_i = (x_{i,1}, \ldots, x_{i,n_i}) \in I_1^{n_i} \), and \( F \) is a quantifier-free Boolean formula with no negations having \( s \) atoms of the kind \( f > 0 \), where \( f \)'s are polynomials with real coefficients of degrees less than \( d \). The cylindrical algebraic decomposition technique from \cite{3, 21} allows to bound from above the number of cells in a representation of \( X \) as a difference between a \( CW \)-complex and its subcomplex. In particular,

\[
b(X) \leq (sd)^{O(n)}. \tag{3.4}
\]

A better upper bound can be obtained as follows. According to \cite{2} (which refines \cite{9, 17}), there exists a Boolean combination

\[
\psi(x_0) = \bigvee_{1 \leq i \leq l} \bigwedge_{1 \leq j \leq I_i} (g_{i,j}(x_0) *_{i,j} 0),
\]

such that \( X = \{\psi(x_0)\} \). Here

\[
*_{i,j} \in \{=, <, >\}, \quad g_{i,j} \in \mathbb{R}[x_0], \quad \text{deg}(g_{i,j}) < dI_{i+1}O(n_i),
\]

\[
I < s(n_0+1)I_{i+1}(n+1)I_{i+1}O(n_i),
\]

\[
J_i < sI_{i+1}(n+1)I_{i+1}O(n_i).
\]

Applying (3.1) to \( X = \{\psi(x_0)\} \), we get

\[
b(X_0) \leq sO(n_0^2I_{i+1}O(n_0))O(n_i) \leq (sd)^{O(n_0^2)O(n_i)} \tag{3.5}
\]

### 4. Basic notation

Let \( X = \overline{X_0} = I_1^{\infty} \setminus X_0 \subset I_1^{\infty} \) be a set defined by a formula (0.1). For example, \( X \) could be a sub-Pfaffian or a semialgebraic set defined by (3.3), where \( F \) is a quantifier-free Boolean formula with no negations. For definiteness assume that \( Q_1 = \exists \) and \( X \) is open in \( I_1^{\infty} \).

Define

\[
X_i := \{(x_0, \ldots, x_i) | Q_{i+1}x_{i+1}Q_{i+2}x_{i+2} \ldots Q_nx_n((x_0, x_1, \ldots, x_n) \in X)\}
\]

for odd \( i \) and

\[
X_i := I_1^{n_0+\ldots+n_i} \setminus \{(x_0, \ldots, x_i) | Q_{i+1}x_{i+1}Q_{i+2}x_{i+2} \ldots Q_nx_n((x_0, x_1, \ldots, x_n) \in X)\}
\]

for even \( i \). Then \( \pi_i(X_i) = \overline{X_{i-1}} \), where \( \pi_i : \mathbb{R}^{n_0+\ldots+n_i} \to \mathbb{R}^{n_0+\ldots+n_{i-1}} \) and tilde denotes the complement in \( I_1^{n_0+\ldots+n_{i-1}} \).

For a set \( I_1^{m_1} \times I_1^{m_{i-1}} \times \ldots \times I_1^{m_1} \) define \( \partial(I_1^{m_1} \times I_1^{m_{i-1}} \times \ldots \times I_1^{m_1}) \) as

\[
(I_1^{m_1} \times I_1^{m_{i-1}} \times \ldots \times I_1^{m_1}) \setminus (I_1^{m_1} \times I_1^{m_{i-1}} \times \ldots \times I_1^{m_1})
\]

for even \( i \) and as the closure of this difference for odd \( i \).
Let $p_1, \ldots, p_i$ be some positive integers to be specified later. Define

$$B_i^j := \partial(I_{v-i}^{n_0 + (p_i + 1)n_1} \times I_{v-i-1}^{(p_i + 1)n_2} \times \cdots \times I_{v-i-n_1-1}^{(p_i - 1)n_1}) \times I_{v-i}^{n_i}.$$  

For any $j, i < j \leq \nu$ define $B_j^i := B_j^i \times I_{v-i}^{n_i}$, where tilde denotes the complement in the appropriate cube.

**Definition 6.**

(i) Let $Y \subset I_{v}^{n_0} \times I_{v-i}^{(p_i + 1)n_1} \times I_{v-i-1}^{(p_i + 1)n_2} \times \cdots \times I_{v-i-n_1-1}^{(p_i - 1)n_1} \times I_{v-i-n_2-1}^{n_2}$, where $1 \leq j \leq i, v \geq j$, and let $J \subset \{(j_1, \ldots, j_i) | 1 \leq j_k \leq p_i + 1, l \leq k \leq i\}$. Then define $\prod_{i,j}^{l+1} Y$ as an intersection of sets

$$\{(x_0, x_1^{(1)}, \ldots, x_i^{(1)}, x_i^{(p_i+1)}) | x_0 \in I_{v}^{n_0}, x_k^{(m)} \in I_{v-k+1}^{n_k} (1 \leq k \leq l-1), x_k^{(m)} \in I_{v-k+1}^{n_k} (l \leq k \leq i), (x_0, x_1^{(1)}, \ldots, x_{i_j}^{(p_i+1)}, x_{i_j}^{(j)}, \ldots, x_i^{(j)}) \in Y\}$$

over all $(j_1, \ldots, j_i) \in J$.

(ii) Let $Y \subset I_{v}^{n_0} \times I_{v-i}^{(p_i + 1)n_1} \times I_{v-i-1}^{(p_i + 1)n_2} \times \cdots \times I_{v-i-n_1-1}^{(p_i - 1)n_1} \times I_{v-i-n_2-1}^{n_2} \times I_{v-i-n_3-1}^{n_3}$. Define $\prod_{i,j}^{l+1} Y$ as an intersection of sets

$$\{(x_0, x_1^{(1)}, \ldots, x_i^{(1)}, \ldots, x_j^{(p_j)}, x_{i_1}^{(j_1)}, \ldots, x_i^{(j)}) \in Y\}$$

over all $(j_1, \ldots, j_i) \in J$.

(iii) If $l = i$ and $J = \{j | 1 \leq j \leq p_i + 1\}$ we use the notation $\prod_{i,j}^{l} Y$ for $\prod_{i,j}^{l+1} Y$.

**Lemma 6. Let**

$$Y \subset I_{v}^{n_0} \times I_{v-i}^{(p_i + 1)n_1} \times I_{v-i-1}^{(p_i + 1)n_2} \times \cdots \times I_{v-i-n_1-1}^{(p_i - 1)n_1} \times I_{v-i-n_2-1}^{n_2} \times I_{v-i-n_3-1}^{n_3}.$$  

*Then for any $J \subset \{(j_1, \ldots, j_i) | 1 \leq j_k \leq p_i + 1, l \leq k \leq i\}$ we have*

$$\prod_{i+1}^{l+1} Y \prod_{i,j}^{l+1} Y = \prod_{i+1}^{l+1} Y \times J Y.$$  

*Proof. Straightforward.*

**Definition 7.** Let $Y, l, i, J$ be as in Definition 6. Define $\bigcup_{i,j}^{l} Y$ and $\bigcup_{i,j}^{l+1} Y$ similar to $\prod_{i,j}^{l} Y$ and $\prod_{i,j}^{l+1} Y$ respectively, replacing in Definition 6 “intersection” by “union”.

**Lemma 7.** (De Morgan law)

$$\bigcup_{i,j}^{l} Y = \left(\prod_{i,j}^{l+1} Y\right)^\sim;$$

$$\bigcup_{i,j}^{l+1} Y = \left(\prod_{i,j}^{l+1} Y\right)^\sim,$$

where tilde denotes complements in the appropriate cubes.
5. Pfaffian case

\textbf{Definition 8.} Let \( t_i = n_0 + n_1(p_1 + 1) + \ldots + n_i(p_i + 1) \). Define projection maps

\[ \pi_i \colon \mathbb{R}^{n_0 + \ldots + n_i} \to \mathbb{R}^{n_0 + \ldots + n_{i-1}} \]

\((x_0, \ldots, x_i) \mapsto (x_0, \ldots, x_{i-1})\),

and for \( j < i \),

\[ \pi_{i,j} \colon \mathbb{R}^{n_j + \ldots + n_i} \to \mathbb{R}^{n_j + \ldots + n_{i-1}} \]

\((x_0, x_1^{(1)}, \ldots, x_j^{(p_j+1)}, x_{j+1}, \ldots, x_i) \mapsto (x_0, x_1^{(1)}, \ldots, x_j^{(p_j+1)}, x_{j+1}, \ldots, x_{i-1})\).

\textbf{Lemma 8.} Let \( Y \subset I_0^{n_0} \times I_{v_1}^{(p_1 n_1)} \times I_{v_2}^{(p_2 n_2)} \times \ldots \times I_{v_{l-1}}^{(p_{l-1} n_{l-1})} \times I_1^{n_1 + \ldots + n_i + n_{i+1}} \).

Then

\[ \bigcup_{i,j} I_{i,j} \pi_{i+1,j-1}(Y) = \pi_{i+1,j} \left( \bigcup_{i,j} I_{i,j} Y \right) \]

\textbf{Proof.} Straightforward.

5. Case of a single quantifier block

According to Theorem 1,

\[ b_{q_0}(X) = b_{q_0}(\overline{X}_0) \leq \sum_{p_1+q_1=q_0} b_{q_1} \left( \prod_{i,j \in J_1^i} X_i \right), \quad (5.1) \]

where \( J_1^i = \{1, \ldots, p_1+1\} \).

Let \( \nu = 1 \), then (3.3) turns into \( \exists x_1 F(x_0, x_1) \), where \( X_1 = \{ F(x_0, x_1) \} \) and \( F(x_0, x_1) \) is a Boolean combination with no negations of \( s \) atomic formulae of the kind \( f > 0 \).

5.1. Polynomial case

Suppose that \( X_1 \) is semialgebraic, with \( f \)'s being polynomials of degrees \( \deg(f) < d \). For any \( k \leq \dim(X) \), we bound the Betti number \( b_k(X) \) from above in the following way. Observe that \( \prod_{i,j \in J_1^i} X_i \) is an open set in \( I_1^{n_0+(p_1+1)n_1} \) definable by a Boolean combination with no negations of \( (p_1+1)s \) atomic formulae of the kind \( g > 0, \deg(g) < d \) in \( t_1 = n_0 + (p_1+1)n_1 \) variables.

According to (3.1), for any \( q_1 \leq \dim(X) \),

\[ b_{q_1} \left( \prod_{i,j \in J_1^i} X_i \right) \leq O(p_1 sd)^{n_0+(p_1+1)n_1} \]

Then due to (5.1), for any \( k \leq \dim(X) \),

\[ b_k(X) \leq \sum_{p_1+q_1=k} O(p_1 sd)^{n_0+(p_1+1)n_1} \leq (ksd)^{O(n_0+kn_1)}. \]

5.2. Pfaffian case

Suppose that \( X_1 \subset I_1^n \) is sub-Pfaffian, with \( f \)'s being Pfaffian functions in an open domain \( G \supset I_1^n \) of order \( \rho \), degree \( (\alpha, \beta) \), having a common Pfaffian chain. Observe
that \( \prod_{1<i,j}^{1} X_1 \) is an open set definable by a Boolean combination with no negations of \((p_1 + 1)s\) atomic formulae of the kind \( g > 0 \), where \( g \) are Pfaffian functions.

Let \( \beta = 1 \) and \( \alpha, \beta \) are Pfaffian functions. According to (3.2), for any \( \nu > 2 \),

\[
\pi_i : \mathbb{R}^{n_0 + \ldots + n_i} \to \mathbb{R}^{n_0 + \ldots + n_{i-1}},
\]

for \( j < i \),

\[
\pi_{i,j} : \mathbb{R}^{n_j + n_{j+1} + \ldots + n_{i-1}} \to \mathbb{R}^{n_j + n_{j+1} + \ldots + n_{i-1}}.
\]

Let \( \nu = 3 \), then the original formula becomes \( \exists x_1 \forall x_2 \exists x_3((x_0, x_1, x_2, x_3) \in X_3) \).

Thereby,

\[
X_1 = \{ \forall x_2 \exists x_3((x_0, x_1, x_2, x_3) \in X_3) \}, \quad X_2 = \{ \exists x_3((x_0, x_1, x_2, x_3) \in X_3) \},
\]

\( X = X_0 \) is open in \( I_1^{n_0} \).

According to Theorem 1,

\[
b_{q_0}(X_0) \leq \sum_{p_1+q_1=q_0} b_{q_1}(\prod_{1<i,j}^{1} X_1).
\]

Applying in succession Lemma 7 (De Morgan law), Lemma 4 (Alexander’s duality), definitions of \( \pi_2 \) and \( \pi_{2,1} \), and Lemma 8 we get

\[
b_{q_1}(\prod_{1<i,j}^{1} X_1) = b_{q_1}(\left( \bigcup_{1<i,j}^{1} X_1 \right)) \leq b_{t_1-q_1-1}\left( \bigcup_{1<i,j}^{1} X_1 \cup \partial I_1^{n_1} \right)
\]

\[
= b_{t_1-q_1-1}\left( \pi_{2,1}\left( \bigcup_{1<i,j}^{1} X_1 \cup \partial I_1^{n_1} \times I_1^{n_2} \right) \right).
\]
Due to Theorem 1, the last expression does not exceed
\[ \sum_{p_2 + q_2 = t_1 - q_1 - 1} b_{q_2} \left( \prod_{j_1}^2 \left( \bigcup_{j_1}^{1,2} X_2 \cup \partial I_1^{n_1} \times I_1^{n_2} \right) \right) = \]
\[ = \sum_{p_2 + q_2 = t_1 - q_1 - 1} b_{q_2} \left( \prod_{j_1}^2 \left( \bigcup_{j_1}^{1,2} X_2 \cup B_2^x \right) \right). \]

In a case of sub-Pfaffian or semialgebraic \( X \) it is now possible to estimate
\[ b_{q_2} \left( \prod_{j_1}^2 \left( \bigcup_{j_1}^{1,2} X_2 \cup B_2^x \right) \right) \]
via the format of \( X_2 \). This completes the description of the case of two quantifier blocks. We now proceed to the case of three blocks.

Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander's duality),
\[ b_{q_2} \left( \prod_{j_1}^2 \left( \bigcup_{j_1}^{1,2} X_2 \cup B_2^x \right) \right) = b_{q_2} \left( \left( \bigcup_{j_1}^2 \left( \prod_{j_1}^{1,2} \tilde{X}_2 \cap \tilde{B}_2^x \right) \right) \right) = (6.1) \]
\[ = b_{t_2 - q_2 - 1} \left( \bigcup_{j_1}^2 \left( \prod_{j_1}^{1,2} \tilde{X}_2 \cap \tilde{B}_2^x \right) \cap \partial (I_1^{n_1} \times I_1^{n_2(p_2 + 1)}) \right). \]

From this point we could have proceeded in a "natural" way similar to the just considered case of two blocks, namely, replacing in the previous expression the set \( \tilde{X}_2 \) by \( \pi_3(X_2) \), then carrying the projection operator to the left to obtain an expression of the kind \( b_{t_2 - q_2 - 1}(\pi_3(\ldots)) \), and after that applying Theorem 1. However, carrying the projection operator through the symbol \( \prod_{j_1}^{1,2} \) (which corresponds to an intersection of some cylindrical sets) would require an introduction of \( p_1 n_2 \) new variables. This would result in a significantly higher upper bound for \( b_{q_2}(X) \). Instead we reduce intersections to unions, then carrying the projection operator to the left does not require new variables.

More precisely, by Lemma 5 (Mayer-Vietoris inequality) expression (6.1) does not exceed
\[ \sum_{1 \leq k_2 \leq p_2 + 1} \sum_{j_2 \subset \{1, \ldots, p_2 + 1\}, |j_2| = k_2} b_{t_2 - q_2 - k_2} \left( \prod_{j_2}^2 \left( \prod_{j_1}^{1,2} \tilde{X}_2 \cap \tilde{B}_2^x \right) \cap \partial (I_1^{n_1} \times I_1^{n_2(p_2 + 1)}) \right). \]

(We estimate a Betti number of the union of cylindrical sets from the definition of the symbol \( \prod_{j_2}^2 \) by a sum of Betti numbers of intersections of various combinations of these sets.)

By Lemma 6,
\[ b_{t_2 - q_2 - k_2} \left( \prod_{j_2}^2 \left( \prod_{j_1}^{1,2} \tilde{X}_2 \cap \tilde{B}_2^x \right) \cap \partial (I_1^{n_1} \times I_1^{n_2(p_2 + 1)}) \right) = \]
\[ = b_{t_2 - q_2 - k_2} \left( \prod_{j_2}^1 \left( \prod_{j_1}^{1,2} \tilde{X}_2 \cap \tilde{B}_2^x \right) \cap \partial (I_1^{n_1} \times I_1^{n_2(p_2 + 1)}) \right), \]

with \( J_2^x = \{1\} \). By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed
\[ \sum_{1 \leq s_* \leq p_2 + k_2 + 1} \sum_{j_2 \subset \{j_1^1 \times j_2, j_2 \subset \{j_1 \times j_2 \}, |j_2| + |j_2^1| = s_*} b_{t_2 - q_2 - k_2 + s_* - 1} \left( \bigcup_{j_2}^1 \tilde{X}_2 \cup \bigcup_{j_2}^2 \tilde{B}_2^x \cup \partial (I_1^{n_1} \times I_1^{n_2(p_2 + 1)}) \right). \]
taking into the account that
\[ \dim \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)}) \right) \leq t_2 \]
and therefore
\[ b_{t_2-q_2-k_2+s_2-1} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)}) \right) = 0 \]
for \( s_2 > q_2 + k_2 + 1 \).

We have
\[ b_{t_2-q_2-k_2+s_2-1} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)}) \right) = \]
\[ b_{t_2-q_2-k_2+s_2-1} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)} \times I_1^{n_3}) \right) = \]
\[ = b_{t_2-q_2-k_2+s_2-1} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)} \times I_1^{n_3}) \right) \]
Due to Theorem 1 the last expression does not exceed
\[ \sum_{i_1+i_0=q_0} b_{i_0} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)} \times I_1^{n_3}) \right) \]
In case of a sub-Pfaffian or a semialgebraic \( X \) it is now possible to estimate
\[ b_{i_0} \left( \bigcup_{2,i}^1 X_2 \cup \bigcup_{2,i}^2 B_2^i \cup \partial (I_2^0 \times I_1^{n_2(p_2+1)} \times I_1^{n_3}) \right) \]
via the format of \( X_3 \).

7. Arbitrary number of quantifiers

**Theorem 2.** For any \( i \) the Betti number \( b_{i_0} (X) \) does not exceed
\[ \sum_{p_1+q_1=q_0} \sum_{p_2+q_2=t_1+1} \sum_{1 \leq k_2 \leq p_2+1} \sum_{J_2^k \subset \{1, \ldots, p_2+1\}, |J_2^k|=k_2} \]
\[ \sum_{1 \leq s_2 \leq q_2+k_2+1} \sum_{J_2^s \subset J_1^s \times J_2^s, |J_2^s|+|J_2^s|=s_2} \sum_{p_3+q_3=t_2-k_2+s_2-1} \]
\[ \ldots \]
\[ \sum_{1 \leq k_{i-1} \leq p_{i-1}+1} \sum_{J_{i-1}^k \subset \{1, \ldots, p_{i-1}+1\}, |J_{i-1}^k|=k_{i-1}} \sum_{1 \leq s_{i-1} \leq q_{i-1}+k_{i-1}+1} \sum_{J_{i-1}^s \subset J_{i-1}^s \times J_{i-1}^s, |J_{i-1}^s|+|J_{i-1}^s|=s_{i-1}} \sum_{p_{i}+q_{i}=t_{i}-q_{i-1}-k_{i-1}+1} \]
\[ b_{i_0} \left( \bigcup_{i}^1 X_i \cup \bigcup_{2 \leq r \leq i} \bigcup_{i-1,J_{i-1}^r} B_{i}^r \cup B_{i}^i \right) \]

**Proof.** Induction on \( i \). Suppose (7.1) is true. Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander’s duality),
\[ b_{i_0} \left( \bigcup_{i}^1 X_i \cup \bigcup_{2 \leq r \leq i} \bigcup_{i-1,J_{i-1}^r} B_{i}^r \cup B_{i}^i \right) = \]
where, by Lemma 6,

\[ \leq b_{r-1,q} \left( \bigcup_i \left( \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \cap \bigcap_{2 \leq r \leq l-1} \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \right) \right) \]

By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

\[ \sum_{1 \leq k_i \leq p_i+1} \sum_{J_i^{j_i} \subset \{1, \ldots, p_i+1\}, |J_i^{j_i}| = k_i} b_{r-1,q} \left( \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \cap \bigcap_{2 \leq r \leq l-1} \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \right) \]

where, by Lemma 6,

\[ = b_{r-1,q} \left( \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \cap \bigcap_{2 \leq r \leq l-1} \prod_{i=1,j_i^{-1} \leq 1} \overline{B_i} \right) \]

where \( J_i^{-1} = \{1\} \). By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

\[ \sum_{1 \leq s_i \leq q_i+k_i+1} \sum_{J_i^{j_i} \subset J_i^{j_i-1} \times J_i^{j_i}, J_i^{j_i-1} \times J_i^{j_i}, |J_i^{j_i}+\ldots+|J_i^{j_i}| = s_i} b_{r-1,q} \left( \bigcup_{i,j_i} \overline{B_i} \cap \bigcap_{2 \leq r \leq l} \prod_{i,j_i} \overline{B_i} \right) \]

We have

\[ = b_{r-1,q} \left( \prod_{i,j_i} \overline{B_i} \cap \bigcap_{2 \leq r \leq l} \prod_{i,j_i} \overline{B_i} \right) \]

\[ \cup \pi_{i+1} (\partial(I_i^{n_0+(p_i+1)n_1} \times \ldots \times I_1^{(p_i+1)n_1})) \]

\[ = b_{r-1,q} \left( \prod_{i,j_i} \overline{B_i} \cap \bigcap_{2 \leq r \leq l} \prod_{i,j_i} \overline{B_i} \right) \]
Due to Theorem 1 the last expression does not exceed
\[
\sum_{p_{i+1}+q_{i+1}=l_{i}=q_i-k_i+s_i-1} b_{q_{i+1}} \left( \prod_{i+1}^{t+1} \left( \bigcup_{i_i,j_i}^{r_i+1} X_{i+1} \cup \bigcup_{2 \leq r \leq i} \bigcup_{i_i,J_i+1}^{r_i+1} B_{i+1}^{r_i} \cup B_{i+1}^{r_i+1} \right) \right).
\]

8. Upper bounds for sub-Pfaffian sets

We first estimate from above the number of additive terms in (7.1). These terms can be partitioned into \(i - 1\) groups of the kind
\[
\sum_{1 \leq k_j \leq p_j+1} \sum_{1 \leq s_j \leq q_i+k_j+1} \sum_{p_j+1+q_j+1=t_j-q_j-k_j+s_j-1} \text{with terms of the kind}
\]
where \(1 \leq j \leq i - 1\).

The number of terms in
\[
\sum_{1 \leq k_j \leq p_j+1} \sum_{1 \leq s_j \leq q_i+k_j+1} \sum_{p_j+1+q_j+1=t_j-q_j-k_j+s_j-1}
\]
is \(2^{p_j+1}\). The number of terms in
\[
\sum_{1 \leq s_j \leq q_i+k_j+1} \sum_{p_j+1+q_j+1=t_j-q_j-k_j+s_j-1}
\]
does not exceed \(2^{j(q_j+k_j+1)}\). The number of terms in
\[
\sum_{1 \leq k_j \leq p_j+1} \sum_{1 \leq s_j \leq q_i+k_j+1} \sum_{p_j+1+q_j+1=t_j-q_j-k_j+s_j-1}
\]
does not exceed \(t_j + 1\).

It follows that the total number of terms in \(j\)th group does not exceed
\[
2^{p_j+1+j(q_j+k_j+1)}(t_j + 1) \leq 2^{O(jt_{j-1})}.
\]

Since \(t_j = n_0+n_1(p_1+1)+\ldots+n_j(p_j+1)\), \(p_i \leq t_{i-1}\), and therefore \(t_j \leq 2^j n_0 n_1 \ldots n_j\), the number of terms in \(j\)th group does not exceed \(2^{O(j^2 n_0 n_1 \ldots n_{j-2})}\). It follows that the total number of terms in (7.1) does not exceed \(2^{O(j^2 n_0 n_1 \ldots n_{j-2})}\).

We now find an upper bound for
\[
b_{q_{i+1}} \left( \prod_{i+1}^{t+1} \left( \bigcup_{i_i,j_i}^{r_i+1} X_{i+1} \cup \bigcup_{2 \leq r \leq i} \bigcup_{i_i,J_i+1}^{r_i+1} B_{i+1}^{r_i} \cup B_{i+1}^{r_i+1} \right) \right).
\]
Assume that \(X_{i+1} = \{ F(x_0, x_1, \ldots, x_n) \} \), where \(F\) is a quantifier-free Boolean formula with no negations having \(s\) atoms of the kind \(f > 0\), \(f\)'s are polynomials or Pfaffian functions of degrees less than \(d\) or \((\alpha, \beta)\) respectively. In Pfaffian case, let
functions \( f \) be defined in an open domain \( G \) by the same Pfaffian chain of order \( \rho \). We assume without loss of generality that \( I_{\nu}^{\rho} \subset G \).

The set \( \bigcup_{1 \leq \nu \leq \nu_1} X_{\nu} \subset \mathbb{R}^{t_{\nu-1}+n_{\nu}} \) is defined by a Boolean formula with no negations having \( |J_{\nu-1}|s \leq s_{\nu-1}s \leq (2t_{\nu-2} + 1)s \) atoms of degrees less than \( d \) (for polynomials) or less than \( (\alpha, \beta) \) (for Pfaffian functions) and at most \( 2t_{\nu-1} + 2n_{\nu} \) linear atoms (defining \( I_{\nu-1}^{t_{\nu-1}+n_{\nu}} \)).

For any \( 2 \leq r \leq \nu \) the set \( B_r^{\nu} \subset \mathbb{R}^{t_{r-1}+n_{r}} \) is defined by a Boolean formula with no negations having \( 4t_{r-1} + 2n_{r} \) linear atomic inequalities. Therefore, all sets of the kind \( B_r^{\nu} \) for \( j \geq r \) are defined by Boolean formulae with no negations having \( 4t_{r-1} + 2(n_{r} + \ldots + n_{j}) \) linear atomic inequalities. In particular, the set \( B_r^{\nu} \subset \mathbb{R}^{t_{r-1}+n_{r}} \) is defined by \( 4t_{r-1} + 2(n_{r} + \ldots + n_{\nu}) \) linear atomic inequalities.

For any \( 2 \leq r \leq \nu - 1 \) the set \( \bigcup_{1 \leq \nu \leq \nu_1} B_r^{\nu} \subset \mathbb{R}^{t_{r-1}+n_{r}} \) is defined by a Boolean formula with no negations having at most \( (4t_{r-1} + 2(n_{r} + \ldots + n_{\nu}))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_{\nu} \) linear atoms.

It follows that the set \( \bigcup_{1 \leq \nu \leq \nu_1} \bigcup_{2 \leq r \leq \nu} B_r^{\nu} \subset \mathbb{R}^{t_{\nu-1}+n_{\nu}} \) is defined by a Boolean formula with no negations having at most \( (4t_{\nu-1} + 2(n_{2} + \ldots + n_{\nu}))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_{\nu} (\nu - 2) \) linear atoms.

The set
\[
\prod_{\nu} \left( \bigcup_{1 \leq \nu \leq \nu_1} X_{\nu} \cup \bigcup_{2 \leq r \leq \nu-1} \bigcup_{2 \leq r \leq \nu} B_r^{\nu} \bigcup B_{\nu}^{\nu} \right) \subset \mathbb{R}^{t_{\nu}}
\] (8.1)
is defined by a Boolean formula with no negations having at most \( (2t_{\nu-2} + 1)s + 2t_{\nu-1} + 2n_{\nu} + (4t_{\nu-1} + 2(n_{2} + \ldots + n_{\nu}))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_{\nu} (\nu - 2)) \) atoms of degrees less than \( d \) for polynomials or less than \( (\alpha, \beta) \) for Pfaffian functions.

Similar calculation shows that, in the Pfaffian case, the set (8.1) is defined by Pfaffian functions having the order at most \( \rho(2t_{\nu-2} + 1)(t_{\nu-1} + 1) \leq O(\rho t_{\nu-2} t_{\nu-1}) \).

### 8.1. Polynomial case

Let functions \( f \) in formula \( F \) be polynomials of degrees \( \deg(f) < d \). Then, according to (3.1),
\[
b_{\nu_0} \left( \prod_{\nu} \left( \bigcup_{1 \leq \nu \leq \nu_1} X_{\nu} \cup \bigcup_{2 \leq r \leq \nu-1} \bigcup_{2 \leq r \leq \nu} B_r^{\nu} \bigcup B_{\nu}^{\nu} \right) \right) \leq O(ds t_{\nu_1} \nu_0 t_{\nu_1} \nu) \leq (2^{\nu} ds n_{0} \ldots n_{\nu-1}) O(2^{\nu} n_{0} n_{1} \ldots n_{\nu})
\]

Using (7.1) in case \( i = \nu \), we get
\[
b_{\nu_0}(X) \leq (2^{\nu_0} ds n_{0} \ldots n_{\nu-1}) O(2^{\nu_0} n_{0} n_{1} \ldots n_{\nu})
\]
(compare with (3.4) and (3.5)).
8.2. Pfaffian case

Let \( f \) be Pfaffian functions in an open domain \( G \supset I_p^q \) of order \( q \), having a common Pfaffian chain. Then, according to (3.2),

\[
b_q \left( \prod_{\nu} \left( \bigcup_{\nu=1}^{\nu(\nu-1)} X_{\nu} \cup \bigcup_{2 \leq \nu \leq q-1} \bigcup_{\nu=1}^{\nu(\nu-1)} B_{\nu} \cup B_{\nu}^{\nu} \right) \right) \leq \\
\leq 2^{O(p^2 t^2 - 2 t^2)} (s t_{\nu-1})^O(t_{\nu}) O(t_{\nu} + \min(t_{\nu}, \rho) \alpha)^O(t_{\nu} + \rho t_{\nu-1}^2 t_{\nu-1}) \leq \\
\leq 2^{O(p^2 t^2 n_1^2 n_2^2 - n_1^2 n_2^2)} O(2^p n_0 n_1 \ldots n_\nu),
\]

\[
(2^p n_0 n_1 \ldots n_\nu (\alpha + \beta))^{O(2^p n_0 n_1 \ldots n_\nu + \rho 2^p n_0^2 n_1^2 n_2^2)}.
\]

Using (7.1) in case \( i = \nu \), we get

\[
b_q(X) \leq 2^{O(v^2 n_0 n_1 \ldots n_\nu + \rho p t^2 n_0 n_1 \ldots n_\nu)} O(2^p n_0 n_1 \ldots n_\nu),
\]

\[
(2^p n_0 n_1 \ldots n_\nu (\alpha + \beta))^{O(2^p n_0 n_1 \ldots n_\nu + \rho 2^p n_0^2 n_1^2 n_2^2)}.
\]

Introducing the notations:

\[
u := 2^p n_0 n_1 \ldots n_\nu, \quad \nu := 2^p n_0^2 n_1^2 \ldots n_\nu^2 n_\nu-1,
\]

we can rewrite this bound in a more compact form

\[
b_q(X) \leq 2^{O(v u + \rho 2^p)} O(2^p + \rho v^2).\]

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References


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