Random Geometric Graphs

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MODELS of RANDOM GRAPHS

Erdos-Renyi $G(n,p)$: start with the complete n-graph, retain each edge with probability p (independently).

Random geometric graph $G(n,r)$. Place n points uniformly at random in $[0,1]^2$. Connect any two points distance at most r apart.

Consider n large and p or r small.

Expected degree of a typical vertex is approximately:

$$\begin{array}{rcl} np & \text{for} & G(n,p) \\ n\pi r^2 & \text{for} & G(n,r) \end{array}$$

Often we choose $p = p_n$ or $r = r_n$ to make this expected degree $\Theta(1)$.

In this case $G(n, r_n)$ resembles the following infinite system:

CONTINUUM PERCOLATION (see Meester and Roy 1996)

Let $\mathcal{P}_\lambda$ be a homogeneous Poisson point process in $\mathbf{R}^d$ with intensity λ, i.e.

$$\mathcal{P}_\lambda(A) \sim \mathrm{Poisson}(\lambda|A|)$$

and $\mathcal{P}_\lambda(A_i)$ are independent variables for $A_1, A_2, \ldots$ disjoint.

Gilbert Graph. Form a graph $\mathcal{G}_\lambda := G(\mathcal{P}_\lambda)$ on $\mathcal{P}_\lambda$ by connecting two Poisson points x, y iff $|x-y| \leq 1$.

Form graph $\mathcal{G}_\lambda^0 := G(\mathcal{P}_\lambda \cup \{\mathbf{0}\})$ similarly on $\mathcal{P}_\lambda \cup \{\mathbf{0}\}$.

Let $p_k(\lambda)$ be the prob. that the component of $\mathcal{G}_\lambda^0$ containing $\mathbf{0}$ has k vertices (also depends on d).

$p_\infty(\lambda) := 1 - \sum_{k=0}^{\infty} p_k(\lambda)$ be the prob. that this component is infinite.

RANDOM GEOMETRIC GRAPHS (rescaled) (Penrose 2003)

Let $U_1, \ldots, U_n$ be independently uniformly randomly scattered in a cube of volume n/λ in d-space. Form a graph $\mathcal{G}_{n,\lambda}$ on $\{1, 2, \ldots, n\}$ by

$$i \sim j \quad \text{iff} \quad |U_i - U_j| \leq 1$$

Define $|C_i|$ to be the order of the component of $\mathcal{G}_{n,\lambda}$ containing i. It can be shown that $n^{-1} \sum_{i=1}^{n} \mathbf{1}\{|C_i| = k\} \xrightarrow{P} p_k(\lambda)$ as $n \to \infty$, i.e. for large n,

$$P\left[n^{-1} \sum_{i=1}^{n} \mathbf{1}\{|C_i| = k\} \approx p_k(\lambda)\right] \approx 1$$

That is, the proportionate number of vertices of $\mathcal{G}_{n,\lambda}$ lying in components of order k, converges to $p_k(\lambda)$ in probability.

A FORMULA FOR $p_k(\lambda)$.

$$p_{k+1}(\lambda) = (k+1)\lambda^k \int_{(\mathbf{R}^d)^k} h(x_1, \ldots, x_k)$$
$$\times \exp(-\lambda A(\mathbf{0}, x_1, \ldots, x_k)) dx_1 \ldots dx_k$$

where $h(x_1, \ldots, x_k)$ is 1 if $G(\{\mathbf{0}, x_1, \ldots, x_k\})$ is connected and $0 \prec x_1 \prec \cdots \prec x_k$ lexicographically, otherwise zero;

and $A(\mathbf{0}, x_1, \ldots, x_k)$ is the volume of the union of 1-balls centred at $0, x_1, \ldots, x_k$.

Not tractable for large k.

THE PHASE TRANSITION

If $p_\infty(\lambda) = 0$, then $\mathcal{G}_\lambda$ has no infinite component, almost surely.

If $p_\infty(\lambda) > 0$, then $\mathcal{G}_\lambda$ has a unique infinite component, almost surely.

Also, $p_\infty(\lambda)$ is nondecreasing in λ.

Fundamental theorem: If $d \geq 2$ then

$$\lambda_c(d) := \sup\{\lambda : p_\infty(\lambda) = 0\} \in (0, \infty).$$

If $d = 1$ then $\lambda_c(d) = \infty$. From now on, assume $d \geq 2$. The value of $\lambda_c(d)$ is not known.

LARGE COMPONENTS FOR THE RGG

Consider again the random geometric graph $\mathcal{G}_{n,\lambda}$ (on n uniform random points in a cube of volume n/λ in d-space)

Let $L_1(\mathcal{G}_{n,\lambda})$ be the size of the largest component, and $L_2(\mathcal{G}_{n,\lambda})$ the size of the second largest component ('size' measured by number of vertices). As $n \to \infty$ with λ fixed (and $\zeta(\lambda)$ as on P16 below):

if $\lambda > \lambda_c$ then $n^{-1} L_1(\mathcal{G}_{n,\lambda}) \xrightarrow{P} p_\infty(\lambda) > 0$

if $\lambda < \lambda_c$ then $(\log n)^{-1} L_1(\mathcal{G}_{n,\lambda}) \xrightarrow{P} 1/\zeta(\lambda)$

and for the Poissonized RGG $\mathcal{G}_{N_n,\lambda}$ ($N_n \sim$ Poisson (n)),

$L_2(\mathcal{G}_{N_n,\lambda}) = O(\log n)^{d/(d-1)}$ in probability if $\lambda > \lambda_c$

CENTRAL AND LOCAL LIMIT THEOREMS. Let $K(\mathcal{G}_{n,\lambda})$ be the number of components of $\mathcal{G}_{n,\lambda}$. As $n \to \infty$ with λ fixed,

$$P\left[\frac{K(\mathcal{G}_{n,\lambda}) - \mathbf{E}K(\mathcal{G}_{n,\lambda})}{\sqrt{n}} \leq t\right] \to \Phi_\sigma(t) := \int_{-\infty}^{t} \varphi_\sigma(x)dx$$

where $\varphi_\sigma(x) := (2\pi\sigma^2)^{-1/2}e^{-x^2/(2\sigma^2)}$ (normal pdf); and if $\lambda > \lambda_c$,

$$P\left[\frac{L_1(\mathcal{G}_{n,\lambda}) - \mathbf{E}L_1(\mathcal{G}_{n,\lambda})}{\sqrt{n}} \leq t\right] \to \Phi_\tau(t).$$

Here σ and τ are positive constants, dependent on λ. Also,

$$\sup_{z\in\mathbf{Z}}\left\{n^{1/2}P[K(\mathcal{G}_{n,\lambda}) = z] - \varphi_\sigma\left(\frac{z - \mathbf{E}K(\mathcal{G}_{n,\lambda}))}{\sqrt{n}}\right)\right\} \to 0.$$

ISOLATED VERTICES. Suppose $d = 2$. Let $N_0(\mathcal{G}_{n,\lambda})$ be the number of isolated vertices. The expected number of isolated vertices satisfies

$$\mathbf{E} N_0(\mathcal{G}_{n,\lambda}) \approx n \exp(-\pi\lambda)$$

so if we fix t and take $\lambda(n) = (\log n + t)/\pi$, then as $n \to \infty$,

$$\mathbf{E} N_0(\mathcal{G}_{n,\lambda(n)}) \to e^{-t}.$$

Also, N_0 is approximately Poisson distributed so

$$P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] \to \exp(-e^{-t}).$$

CONNECTIVITY. Note $\mathcal{G}_{n,\lambda}$ is connected iff $K(\mathcal{G}_{n,\lambda}) = 1$.

Clearly $P[K(\mathcal{G}_{n,\lambda}) = 1] \leq P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0]$. Again taking $\lambda(n) = (\log n + t)/\pi$ with t fixed, it turns out (Penrose 1997) that

$$\lim_{n\to\infty} P[K(\mathcal{G}_{n,\lambda(n)}) = 1] = \lim_{n\to\infty} P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] = \exp(-e^{-t})$$

or in other words,

$$\lim_{n\to\infty} (P[K(\mathcal{G}_{n,\lambda(n)}) > 1] - P[N_0(\mathcal{G}_{n,\lambda(n)}) > 0]) = 0.$$

THE CONNECTIVITY THRESHOLD

Let $V_1, \ldots, V_n$ be independently uniformly randomly scattered in $[0,1]^d$. Form a graph $\mathcal{G}_n^r$ on $\{1, 2, \ldots, n\}$ by

$$i \sim j \quad \text{iff} \quad |V_i - V_j| \le r.$$

Given the values of $V_1, \ldots, V_n$, define the connectivity threshold $\rho_n(K = 1)$, and the no-isolated-vertex threshold $\rho_n(N_0 = 0)$, by

$$\rho_n(K = 1) = \min\{r : K(\mathcal{G}_n^r) = 1\};$$
$$\rho_n(N_0 = 0) = \min\{r : N_0(\mathcal{G}_n^r) = 1\}.$$

The preceding result can be interpreted as giving the limiting distributions of these thresholds (suitably scaled and centred) as $n \to \infty$: they have the same limiting behaviour.

Taking $\lambda(n) = (\log n + t)/\pi$ with t fixed, we have

$$P[K(\mathcal{G}_{n,\lambda(n)}) = 1] = P[K(\mathcal{G}_n^{\sqrt{\lambda(n)/n}}) = 1]$$
$$= P\left[\rho_n(K = 1) \leq \sqrt{(\log n + t)/(\pi n)}\right]$$

so the earlier result

$$\lim_{n\to\infty} P[K(\mathcal{G}_{n,\lambda(n)}) = 1] = \lim_{n\to\infty} P[N_0(\mathcal{G}_{n,\lambda(n)}) = 0] = \exp(-e^{-t})$$

implies

$$\lim_{n\to\infty} P[n\pi(\rho_n(K = 1))^2 - \log n \leq t] = \exp(e^{-t})$$

and likewise for $\rho_n(N_0) = 0$. In fact we have a stronger result:

$$\lim_{n\to\infty} P[\rho_n(K = 1) = \rho_n(N_0 = 0)] = 1.$$

MULTIPLE CONNECTIVITY. Given $k \in \mathbf{N}$, a graph is k-connected if for any two distinct vertices there are k disjoint paths connecting them.

Let $\rho_{n,k}$ be the smallest r such that G_n^r is k-connected.

Let $\rho_n(N_{<k} = 0)$ be the smallest r such that $\mathcal{G}_n^r$ has no vertex of degree less than k (a random variable determined by $V_1, \ldots, V_n$). Then

$$\lim_{n\to\infty} P[\rho_{n,k} = \rho_n(N_{<k} = 0)] = 1.$$

The limit distribution of $\rho_n(N_{<k} = 0)$ can be determined via Poisson approximation, as with $\rho_n(N_0 = 0)$.

HAMILTONIAN PATHS

Let $\rho_n(\mathrm{Ham})$ be the smallest r such that $\mathcal{G}_n^r$ has a Hamiltonian path (i.e. a self-avoiding tour through all the vertices). Clearly

$$\rho_n(\mathrm{Ham}) \geq \rho_n(N_{<2} = 0).$$

In fact (Balogh, Bollobas, Walters, Krivelevich, Müller 2009),

$$\lim_{n\to\infty} P[\rho_n(\mathrm{Ham}) = \rho_n(N_{<2} = 0)] = 1.$$

CONTINUITY OF $p_\infty(\lambda)$

Clearly $p_\infty(\lambda) = 0$ for $\lambda < \lambda_c$

Also $p_\infty(\lambda)$ is increasing in λ on $\lambda > \lambda_c$.

Less trivially, it is known that $p_\infty(\lambda)$ is continuous in λ on $\lambda \in (\lambda_c, \infty)$ and is *right* continuous at $\lambda = \lambda_c$, i.e.

$$p_\infty(\lambda_c) = \lim_{\lambda \downarrow \lambda_c} p_\infty(\lambda).$$

So $p_\infty(\cdot)$ is continuous on $(0, \infty)$ iff

$$p_\infty(\lambda_c) = 0.$$

This is known to hold for $d = 2$ (Alexander 1996) and for large d (Tanemura 1996). It is conjectured to hold for all d.

LARGE-k ASYMPTOTICS FOR $p_k(\lambda)$

Suppose $\lambda < \lambda_c$. Then there exists $\zeta(\lambda) > 0$ such that

$$\zeta(\lambda) = \lim_{n\to\infty} \left(-n^{-1} \log p_n(\lambda)\right)$$

or more informally, $p_n(\lambda) \sim (e^{-\zeta(\lambda)})^n$.

Proof uses subadditivity. With $\mathbf{x}_1^n := (x_1, \ldots, x_n)$, recall

$$p_{n+1}(\lambda) = (n+1)\lambda^n \int h(\mathbf{x}_1^n) e^{-\lambda A(\mathbf{0},\mathbf{x}_1^n)} d\mathbf{x}_1^n.$$

Setting $q_n := p_{n+1}/(n+1)$, can show $q_n q_m \leq q_{n+m-1}$, so $-\log q_n/(n-1) \to \inf_{n\geq 1}(-\log q_n/(n-1)) := \zeta$ as $n \to \infty$.

That $\zeta(\lambda) > 0$ is a deeper result.

LARGE-k ASYMPTOTICS: THE SUPERCRITICAL CASE

Suppose $\lambda > \lambda_c$. Then

$$\limsup_{n\to\infty} \left(n^{-(d-1)/d} \log p_n(\lambda) \right) < 0$$

$$\liminf_{n\to\infty} \left(n^{-(d-1)/d} \log p_n(\lambda) \right) > -\infty$$

Loosely speaking, this says that in the supercritical case $p_n(\lambda)$ decays exponentially in $n^{1-1/d}$, whereas in the subcritical case it decays exponentially in n.

OTHER PROPERTIES OF RGGS

Asymptotic behaviour of other quantities arising from of random geometric graph have been considered, including

The largest and smallest degree.

The clique and chromatic number.

In some cases, non-uniform distributions of the vertices $V_1, \ldots, V_n$ have been considered.

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