Random parking, sequential adsorption, and the jamming limit

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Abstract

Identical cars are dropped sequentially from above into a large parking lot. Each car is positioned uniformly at random, subject to non-overlap with its predecessors, until jamming occurs. There have been many studies of the limiting mean coverage as the parking lot becomes large, but no complete proof that such a limit exists, until now.

We prove spatial laws of large numbers demonstrating that for various multidimensional random and cooperative sequential adsorption schemes such as the one above, the jamming limit coverage is well-defined.

1 Introduction

The classical random car-parking model of Rényi (1958) goes as follows. Unit-length cars arrive sequentially at a roadside of of length L. Each car parks at a position along the length of the kerb chosen uniformly at random subject to the obvious constraint of non-overlap with previously parked cars, until there are no available spaces left of length greater than 1 ('jamming' of the interval $[0, L]$). With the resulting (random) number of cars at jamming denoted $N(L)$, Rényi proved that

$$\lim_{L\to\infty} \frac{\mathbb{E}\, N(L)}{L} = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u} du\right) dt \approx 0.748. \tag{1.1}$$

The model readily generalizes to two or more dimensions. Indeed, Rényi's one-dimensional analysis was motivated by analogy to the less mathematically tractable question of physical sphere-packing in three dimensions. We next describe the analogous two-dimensional model as formulated by Palásti (1960).

Let $A = A(s,t) = [0,s] \times [0,t]$ be a target region in $\mathbb{R}^2$, and let D be a specified bounded open set in $\mathbb{R}^2$ containing the origin (for example, a square or disk). Suppose items are placed sequentially at random in the region A. Assume each successive item is a translate of D with location distributed uniformly at random over A, subject to the constraint of non-overlap with previously placed items. The process terminates when there is no available space left large enough to contain a new item (jamming of A). Let

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$N(A;A)$ denote the (random) number of items packed in at the termination time (the second A in the notation is not a misprint; its reason will become apparent later. If there is a third A, then it *is* a misprint). Our main interest here is in $N(A;A)$.

The two-dimensional model is a version of irreversible *random sequential adsorption* (RSA) which has become very important in the physical and biological sciences, particularly in the study of the deposition of colloidal particles or proteins on a surface. See Evans (1993) for a comprehensive survey. Other surveys include Bartelt and Privman (1991), Privman (2000), Senger *et al.* (2000), Talbot *et al.* (2000) and Wang (2000). A key feature is the irreversibility of the deposition of items, which distinguishes RSA from equilibrium models such as Gibbs distributions. In the statistical literature, an identical model is referred to as *simple sequential inhibition* (SSI); see Diggle (1983) or Stoyan Kendall and Mecke (1995). In the modeling of scheduling protocols in operations research, the model is known as *on-line packing*; see for example Coffman *et al.* (1998). Other processes for which parking-type models have been used include rock fragmentation and election results; see Evans (1993), page 1312.

For further results on the Rényi (1958) model and its variants, see for example Ney (1962), Dvoretzky and Robbins (1964), Mannion (1979), Coffman *et al.* (1998), Itoh and Shepp (1999). In all of these developments, the methods used seem to be strictly one-dimensional, and in higher dimensions there have been many simulation studies but not much in the way of rigorous theory. See Solomon and Weiner (1986) for a survey of developments up to that time.

Palásti (1960) made an early study of the two-dimensional model described above, and made two conjectures that will concern us here. The first conjecture is that if we write $M(s,t)$ for $\mathbb{E}\,N(A;A)$, with $A = A(s,t)$, then

$$\lambda = \lambda(D) := \lim_{s,t\to\infty} \frac{M(s,t)}{st} \quad \text{exists.} \tag{1.2}$$

The second conjecture is that in the special case where D is a unit square, the limit λ is equal to the square of the Rényi limit given in (1.1). Both conjectures readily generalize to higher dimensions. Palásti gives a partial proof of the first conjecture, which is incomplete because it relies on a plausible but unproved hypothesis. She bases her second conjecture on an 'heuristic argument' which she does not provide.

If the set D is taken to have unit area, the putative limiting value λ of $\mathbb{E}\,M(s,t)/(st)$ is sometimes known as the 'jamming limit coverage', and has been the subject of numerous simulation studies for various sets D. For general accounts of these, see Wang (2000), section F of Evans (1993), Solomon and Weiner (1986). The jamming limit coverage has been estimated by simulations as roughly around 0.547, when D is a disk, and as roughly around 0.562 when D is a square (See Evans (1993), pp. 1293 and 1312). The latter figure indicates that Palásti's second conjecture is most likely false.

Many variants of the basic model have also been proposed, mainly in the RSA literature in the physical sciences. These include lattice models, models with more than one type of shape, models with some kind of longer-range interaction besides hard-core repulsion (known as *cooperative sequential adsorption* or CSA), and so forth. The extent of activity on these models is indicated by more than 200 citations to date for Evans (1993). Typical questions are concerned with the value of the jamming limit coverage, the rate of approach to jamming, the spatial distribution of items in the jamming limit, and the fluctuations of $N(A;A)$.

A good deal of this activity is to some extent founded on the assumption that Palásti's first conjecture (1.2) is true. However, it seems that this has never been proved. In fact, Solomon and Weiner (1986, p. 2595) assert that the experimental evidence suggests the first conjecture is probably false.

Our main purpose is to prove that on the contrary, (1.2) is true. In fact we prove a law of large numbers (LLN) which says that $N(A)/|A|$ converges in mean to a constant λ, a stronger result than (1.2). We also allow for more general target sets than rectangles. The result is given in Theorem 2.1 below. The method of proof works in any dimension, and is applicable in modified form to many variants of the basic model, some of which we consider later on.

As well as the total number of points, one might be concerned with more detailed information about the spatial distribution of adsorbed items. We shall present a secondary result (Theorem 2.2), which says, loosely speaking, that the local distribution of adsorbed points stabilizes as the target region becomes large.

With a LLN established for $N(A;A)$, it is natural to look for an associated central limit theorem (CLT) to show $N(A;A)$ is approximately normally distributed. We hope to investigate this in future work.

The remainder of this paper is organised as follows. Section 2 contains the formal statement of our main results as just described, and an outline of the proof and its relation to previous work on the jamming limit. Details of the proof are given in Sections 3 and 4. In Sections 5, 6, and 7 we extend the result to non-uniform distributions of locations of incoming items, to packings of items with randomly varying shapes or spins, and to a CSA model, respectively.

Notation: for a set S in $\mathbb{R}^d$, we shall say S is *nonnull* if its Lebesgue measure is strictly positive. We shall write $|S|$ to mean either the number of elements of S (if S is finite) or the Lebesgue measure of S (if S is nonnull). Let $\mathbf{0}$ denote the origin $(0,\dots,0)$ of $\mathbb{R}^d$.

For $K > 0$ and $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ we define the 'box' $B_K(x) \subset \mathbb{R}^d$ by

$$B_K(x) = \prod_{i=1}^{d} [x_i - K, x_i + K].$$

Also set $\|x\| = \max_i |x_i|$, the l_∞ (maximum-component) norm of x.

For any sequence of sets $(A_n)_{n\geq 1}$ we make the definition

$$\liminf(A_n) := \cup_{n\geq 1} \cap_{m\geq n} A_m.$$

Given a bounded set $A \subset \mathbb{R}^d$, let ∂A be the intersection of the closure A with that of its complement, and for $r > 0$, let $\partial_r A$ denote the set of all $x \in \mathbb{R}^d$ lying at a Euclidean distance at most r from ∂A. If L is a subset of $\mathbb{Z}^d$, let $\partial^{\mathbb{Z}} L$ be the set of elements of L at a Euclidean distance at most 1 from $\mathbb{Z}^d \setminus L$.

If E is an event in a given probability space let $\mathbf{1}_E$ be the indicator random variable taking the value 1 if E occurs and 0 if not.

2 Basic results

In this section we present and discuss our main results on the Palásti model of RSA. The model is as already described, except that from now on we work in $\mathbb{R}^d$ rather than $\mathbb{R}^2$ to emphasise that the arguments work in any dimension d. Also, the target region A, previously assumed to be a rectangle, is now allowed to be an arbitrary bounded Borel nonnull set in $\mathbb{R}^d$.

Theorem 2.1 *There is a constant $\lambda = \lambda(d, D) > 0$ such that if $(A_n)_{n\geq 1}$ is a sequence of bounded nonnull Borel subsets of $\mathbb{R}^d$ satisfying $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$, then for any $p \in [1, \infty)$,*

$$\frac{N(A_n; A_n)}{|A_n|} \xrightarrow{L^p} \lambda. \tag{2.3}$$

This result is a law of large numbers for $N(A; A)$. Later we shall briefly discuss an associated strong law of large numbers.

The reader might guess that a result like Theorem 2.1 can be proved by subadditivity-type methods, but it is not clear whether any form subadditivity holds for $N(A; A)$ or its mean, even in one dimension. The unproved hypothesis which forms the basis of Palásti's original proof of (1.2) is essentially a form of subadditivity.

To proceed further, we need a few notions from point process theory. Let $\mathcal{S}$ be the space of locally finite subsets of $\mathbb{R}^d$. For $\xi \in \mathcal{S}$ and $B \subset \mathbb{R}^d$, let $N_\xi(B)$ denote the number of elements of ξ in B (so $N_\xi(\cdot)$ is a counting measure). A *point process* on $\mathbb{R}^d$ is a random element ξ of $\mathcal{S}$. For more details, see for example Daley and Vere-Jones (1988) Stoyan, Kendall and Mecke (1995), or Resnick (1987).

A 'soft' argument for the existence of a form of jamming limit coverage goes as follows. Consider a Markov process $(\xi_t, t \geq 0)$ taking values in $\mathcal{S}$, in which points, once added, are never removed. Assume the probability of adding a point in the volume element $dxdt$ ($x \in \mathbb{R}^d, t \in [0, \infty)$) is equal to $dxdt$ if there is no overlap of the translate of D centred at x with translates of D centred at existing points just prior to time t, and is equal to zero otherwise. Assume ξ_0 is the empty set. Since the dynamics are spatially homogeneous, for each t, ξ_t is a stationary point process on the whole of $\mathbb{R}^d$ (meaning that at each time t, the spatial distribution of points is invariant under space-translation). By general properties of stationary point processes the point process ξ_t has a density λ_t (the mean number of points per unit area), which is clearly bounded and nondecreasing in t, so must converge to a limit λ' as $t \to \infty$.

Two objections might be made to the above argument. The first objection is that the existence and uniqueness of a Markov process with the above dynamics is not immediately clear. This issue was discussed in Evans (1993), page 1302, but only for lattice RSA models. The second objection is that the soft argument does not take us much closer to knowing whether the limit λ in (1.2) exists, and if it does, whether it is equal to the limit λ' given by the soft argument. Typically, a physical system exhibiting RSA or a computer simulation thereof will be on a large but finite target region, so that it is λ, not λ', that is arguably of greater interest.

Here is an outline of the proof of Theorem 2.1. We use the following alternative definition of $N(A; A)$. Items arrive sequentially with locations uniformly distributed over A, and an item is accepted unless it overlaps a previously accepted item, in which case it is rejected. Let $N(A; A)$ denote the ultimate number of accepted items. Clearly, this is equivalent to the earlier definition of $N(A; A)$. It useful to assume that the locations in space-time of arrivals form a homogeneous Poisson point process in $A \times [0, \infty)$. Let ξ_t^A denote the set of spatial locations of accepted Poisson arrivals up to time t. We can keep track of the accepted points via a graphical representation on the Poisson points, similar to the graphical representations long used in interacting particle systems theory on the lattice (see e.g. Durrett (1988)). Extending the Poisson process and the graphical representation to the whole of space-time, we obtain for each t a stationary point process ξ_t of locations of items accepted up to time t, which matches the informal description of ξ_t

given in the soft argument above. The graphical representation enables us to estimate the difference between ξ_t^A and ξ_t (essentially, a boundary effect) via a comparison a stochastic spatial growth model (first passage percolation; see Durrett (1988)), and using a counting argument to bound the growth rate. In effect, this enables us to fill in the gaps in the soft argument. Convergence of means is strengthened to convergence in p-th moment by an application of the Ergodic Theorem to the stationary point processes ξ_t and ξ.

The result shows that a simulation of the Palásti RSA process on a square will, in principle, yield a coverage that approximates with high probability to the quantity of interest λ, which is the same as λ' described above. A number of simulations in the literature have in fact been on the *torus*: it is possible to show that for these, too, yield a coverage which approximates to λ; loosely speaking, boundary conditions are not too important. However, we shall not go into details in the case of the torus.

Much of the literature from the physical sciences on RSA processes is effectively concerned with the point processes ξ_t and the limiting point process $\xi := \cup_{t>0}\xi_t$. As well as their densities, there is interest in quantities such as correlations and n-point density functions for these point processes; see for example Talbot *et al.* (2000). As in the case of the jamming limit coverage, it seems worthwhile to assure ourselves that such analyses of the infinite-space system are a good approximation to the system on a sufficiently large bounded target region. Therefore we give a second result which says that the distribution of points, for the Palásti RSA process on a bounded target set A, does indeed converge locally to the distribution of points generated by an RSA process in the whole of $\mathbb{R}^d$.

If ζ and ζ_n ($n \in \mathbb{N}$) are point processes on $\mathbb{R}^d$, we say the sequence ζ_n converges weakly to ζ if the finite-dimensional distributions converge, i.e. if for any finite collection of bounded Borel sets B_i satisfying $N_\zeta(\partial B_i) = 0$ almost surely, the joint probability distributions of $N_{\zeta_n}(B_i)$ converge weakly to those of $N_\zeta(B_i)$. This is equivalent to various other definitions of weak convergence; see e.g. Daley and Vere-Jones (1988), section 9.1. Our next result concerns convergence of the point processes ξ_t^A already alluded to, and of the limiting point process ξ^A of all points ultimately accepted in the target set A, as A becomes large.

Theorem 2.2 *There exist point processes ξ and $\xi_t, t > 0$, such that if $(A_n)_{n\geq 1}$ is any sequence of target sets with $\liminf_{n\to\infty} A_n = \mathbb{R}^d$, the sequence of point processes ξ^{A_n} converges weakly to ξ, and for all $t > 0$, the sequence of point processes $\xi_t^{A_n}$ converges weakly to ξ_t.*

3 A graphical construction

In this section we describe a graphical construction which can be used to generate Palásti's RSA process both on a bounded target set A, and on the unbounded target set $\mathbb{R}^d$. This construction will gives us a rigorous demonstration of existence of the point process ξ_t alluded to in the previous section. The model is generated by an input of random translates of a specified bounded open set $D \subset \mathbb{R}^d$ with $0 \in D$ (possibly a ball). On occasion we may refer to the set $x + D := \{x + y : y \in D\}$ as *the translate of D centred at x*.

Assume that on an underlying probability space we have a homogeneous Poisson process $\mathcal{P}$ of unit rate on $\mathbb{R}^d \times [0, \infty)$. Given a bounded nonnull Borel target set $A \subset \mathbb{R}^d$, label the points of the restriction of $\mathcal{P}$ to $A \times [0, \infty)$ as $\{(X_i, T_i)\}_{i=1}^{\infty}$ with $T_1 < T_2 < T_3 < \cdots$. Recursively for $i = 1, 2, 3, \ldots$, let item i be accepted if $(X_i + D) \cap (X_j + D) = \emptyset$

for all $j < i$ such that j was accepted. Let $I(X_i, T_i; A)$ be equal to 1 if item (X_i, T_i) is accepted, and zero otherwise. For Borel $B \subset \mathbb{R}^d$ and for $t > 0$ define

$$N_t(B; A) = \sum_{i=1}^{\infty} I(X_i, T_i; A)\mathbf{1}_{\{T_i \le t, X_i \in B\}}; \qquad N(B; A) = \lim_{t \to \infty} N_t(B; A). \tag{3.1}$$

Then $N(A; A)$ is clearly the total number of accepted items in a realization of Palásti's model as described at the start of this article.

By an elementary rescaling argument, it suffices to prove Theorems 2.1 and 2.2 under the extra assumption that D is small enough to satisfy

$$D \cap (y + D) = \emptyset \ \text{ for all } y \in \mathbb{R}^d \text{ with } \|y\| \ge 1, \tag{3.2}$$

and from now on we assume (3.2).

An *oriented graph* is a special kind of directed graph in which there is no pair of vertices $\{x, y\}$ for which both (x, y) and (y, x) are included as directed edges. By a *circuit* in an oriented graph we shall mean an oriented path that ends up where it started. We shall say that x is a *parent* of y and y is an *offspring* of x if there is an oriented edge from x to y. By a *root* of an oriented graph we mean a vertex with no parents.

The graphical construction goes as follows. Make the points of the Poisson process $\mathcal{P}$ on $\mathbb{R}^d \times [0, \infty)$ into the vertices of an infinite oriented graph, denoted $\mathcal{G}$, by putting in an oriented edge $(X, T) \to (X', T')$ whenever $(X' + D) \cap (X + D) \neq \emptyset$ and $T < T'$. For completeness we also put an edge $(X, T) \to (X', T')$ whenever $(X' + D) \cap (X + D) \neq \emptyset$, $T = T'$, and X precedes X' in the lexicographic ordering on $\mathbb{R}^d$, but in practice the probability that $\mathcal{P}$ induces this second kind of edge is zero.

It can be useful to think of the oriented graph as representing the spread of an 'epidemic' through space over time; each time an individual is 'born' at a Poisson point in space-time, it becomes (and stays) infected if there is an earlier infected point nearby in space (in the sense that the translates of D centred at the two points overlap). This graph determines which items are to be accepted. Before describing how this works, we shall give a key lemma providing a bound for the rate of spread of the epidemic.

Divide $\mathbb{R}^d$ into unit cubes, with the centres of the cubes at the points of the integer lattice $\mathbb{Z}^d$. For $z \in \mathbb{Z}^d$ let Q_z be the cube centred at z. By (3.2), if there is an edge of $\mathcal{G}$ from (X, T) to (Y, U), then if Q_x is the cube containing X and Q_y the cube containing y, then $\|x - y\| \le 1$, so the cubes Q_x and Q_y are either the same or adjacent (possibly diagonally), and $\|x - y\| \le 1$.

For $x, y \in \mathbb{Z}^d$, let us say that *y is affected by x before time t* if there exists a (directed) path in the oriented graph that starts at some Poisson point (X, T) with $X \in Q_x$, and ends at some Poisson point (Y, U) with $Y \in Q_y$ and $U \le t$. In terms of the 'epidemic', y is affected by x before time t if, on the assumption that all particles born in Q_x are infected, the epidemic spreads to Q_y by time t. Define the event $E_t(x, y)$ by

$$E_t(x, y) = \{y \ \text{ is affected by } x \text{ before time } t\}. \tag{3.3}$$

For x and y in $\mathbb{Z}^d$, we define a *path* γ from x to y (written $\gamma : x \rightsquigarrow y$) to be any sequence $x_0 = x, x_1, x_2, \ldots, x_n = y$ of distinct elements of $\mathbb{Z}^d$, with $\|x_i - x_{i-1}\| = 1$ for each $i = 1, 2, \ldots, n$. For such a path we write $|\gamma| = n$.

Given $\gamma = (x_0, \ldots, x_n)$, let S_γ be the time it would take for the infection to pass from x to y along the path γ, if only infections along that path were allowed. More formally, set $S_{\gamma,0} = 0$ and recursively define random variables $S_{\gamma,j}$ for $j = 1, 2, \ldots, n$ by

$$S_{\gamma,j} := S_{\gamma,j-1} + W_{\gamma,j},$$

where $W_{\gamma,j}$ is the time from $S_{\gamma,j-1}$ to time of the next Poisson arrival in the cube Q_{x_j}. Then set $S_\gamma = S_{\gamma,n}$. Then

$$S_\gamma = \sum_{j=1}^{n} W_{\gamma,j},$$

a sum of independent exponential variables with parameter 1. The following key result says that the rate of spread of the epidemic is at most linear with high probability, and is proved using an argument adapted from FPP (see for example Durrett (1988)). Recall that $\mathbf{0}$ denotes the origin of $\mathbb{Z}^d$.

Lemma 3.1 *There is a constant $\delta_1 \in (0,1)$ such that for all $r > 0$, the probability that $\mathbf{0}$ is affected from outside $[-r,r]^d$ before time $\delta_1 r$ is less than $2(3^{-r})$, or in other words,*

$$P\left[\cup_{y\in\mathbb{Z}^d\setminus[-r,r]^d} E_{\delta_1 r}(y,\mathbf{0})\right] \le 2(3^{-r}). \tag{3.4}$$

Proof. By Boole's inequality, for $r, \delta > 0$ we have

$$\begin{aligned} P\left[\cup_{y\in\mathbb{Z}^d\setminus[-r,r]^d} E_{\delta r}(y,\mathbf{0})\right] &\le \sum_{\gamma\rightsquigarrow\mathbf{0},|\gamma|\ge r} P[S_\gamma \le \delta r] \\ &\le \sum_{\gamma\rightsquigarrow 0,|\gamma|\ge r} P[S_\gamma \le \delta|\gamma|]. \end{aligned}$$

Let W denote an exponential random variable with mean 1. For $\theta > 0$, by Markov's inequality we have

$$P[S_\gamma \le \delta|\gamma|] \le \frac{\mathbb{E}[e^{-\theta S_\gamma}]}{e^{-\theta\delta|\gamma|}} = \left(e^{\delta\theta}\mathbb{E}[e^{-\theta W}]\right)^{|\gamma|}.$$

Let $\alpha = 3^{-(d+1)}$. Take $\theta > 0$ with $\mathbb{E}[e^{-\theta W}] < \alpha/2$ and δ_1 with $e^{\delta_1\theta} < 2$. Then $P[S_\gamma \le \delta_1|\gamma|] \le \alpha^{|\gamma|}$. Since each lattice point has $3^d - 1$ neighbours, the number of paths of length n ending at the origin is at most 3^{dn}, so

$$\begin{aligned} P\left[\cup_{y\in\mathbb{Z}^d\setminus[-r,r]^d} E_{\delta_1 r}(y,\mathbf{0})\right] &\le \sum_{n\ge r} \sum_{\gamma\rightsquigarrow\mathbf{0}:|\gamma|=n} P[S_\gamma \le \delta_1 n] \\ &\le \sum_{n\ge r} 3^{dn}\alpha^n \le 2(3^{-r}). \qquad \blacksquare \end{aligned}$$

Corollary 3.1 *Let $z \in \mathbb{Z}^d$ and $t > 0$. With probability 1, z is affected before time t by at most finitely many y.*

Proof. It suffices to consider the case $z = \mathbf{0}$. Let $r > t/\delta_1$. The probability that $\mathbf{0}$ is affected before time t by infinitely many y is bounded by the probability that $\mathbf{0}$ is affected before time t by some y outside $[-r,r]$, and therefore is at most $2(3^{-r})$. Since r can be arbitrarily large, the result follows. $\blacksquare$

The method of reconstructing the set of accepted items from the graph $\mathcal{G}$ goes as follows. Let $A \subset \mathbb{R}^d$ be a (possibly unbounded) Borel set in $\mathbb{R}^d$, and let $\mathcal{P}_A$ denote the set $\mathcal{P} \cap (A \times [0,\infty))$, i.e. the set of Poisson points that lie in $A \times [0,\infty)$. Let $\mathcal{G}|_A$ denote the restriction of $\mathcal{G}$ to the vertex set $\mathcal{P}_A$. Recursively define subsets $F_i(A), G_i(A), H_i(A)$ of $\mathcal{P}_A$, $i = 1,2,3,\ldots$ as follows.

Let $F_1(A)$ be the set of roots of the oriented graph $\mathcal{G}|_A$, and let $G_1(A)$ be the set of offspring of roots. Set $H_1(A) = F_1(A) \cup G_1(A)$. For the next step, remove the set $H_1(A)$ from the vertex set, and define $F_2(A)$ and $G_2(A)$ in the same way; so $F_2(A)$ is the set of roots of the restriction of $\mathcal{G}$ to vertices in $\mathcal{P}_A \setminus H_1(A)$, and $G_2(A)$ is the set of vertices in $\mathcal{P}_A \setminus H_1(A)$ that are offspring of vertices in $F_2(A)$. Set $H_2(A) = F_2(A) \cup G_2(A)$, remove the set $H_2(A)$ from $\mathcal{P}_A \setminus H_1(A)$ and repeat the process to obtain $F_3(A), G_3(A), H_3(A)$. Continuing *ad infinitum* gives us subsets $F_i(A), G_i(A)$ of $\mathcal{P}_A$ defined for $i = 1, 2, 3, \ldots$. It is clear from the construction that these sets are disjoint. If at some stage we run out of roots then all but finitely many of the sets $F_i(A), G_i(A)$ will be empty.

In the special case where the initial set A is taken to be the whole of $\mathbb{R}^d$, let us make the abbreviations F_i for $F_i(\mathbb{R}^d)$ and G_i for $G_i(\mathbb{R}^d)$. Thus F_1 is the set of all roots in the whole of $\mathcal{G}$, G_1 is the set of all offspring in $\mathcal{G}$ of vertices in F_1, and so on. As before, we obtain an infinite sequence of disjoint subsets $F_1, G_1, F_2, G_2, \ldots$ of $\mathcal{P}$.

Lemma 3.2 *Let A be a bounded nonnull Borel set in $\mathbb{R}^d$. With probability 1, the sets $F_1(A), G_1(A), F_2(A), G_2(A), \ldots$ form a partition of $\mathcal{P}_A$, and the sets $F_1, G_1, F_2, G_2, \ldots$ form a partition of $\mathcal{P}$.*

Moreover, if A is the target set for the Palásti RSA model, the set of accepted items is the union $\cup_{i=1}^{\infty} F_i(A)$.

Proof. First we show that every element of $\mathcal{P}_A$ is assigned to one of the sets $F_i(A)$, $G_i(A)$, so that these form a partition of A.

Let $t > 0$ and set $\mathcal{P}_{A,t} = \mathcal{P} \cap (A \times [0, t])$. If the (random but almost surely finite) number of points of $\mathcal{P}_{A,t}$ is N, then every point of $\mathcal{P}_{A,t}$ is assigned to one of the sets $F_1(A), G_1(A), \ldots, F_N(A), G_N(A)$. This is because at each iteration of the algorithm defining the sets $F_i(A), G_i(A)$, the set of hitherto unassigned points of $\mathcal{P}_{A,t}$, if nonempty, contains at least one root, so at least one more element of $\mathcal{P}_{A,t}$ gets assigned at each successive iteration, until all elements of $\mathcal{P}_{A,t}$ are assigned, which must happen in at most N iterations.

Since the above argument applies with probability 1 for all t, it follows that with probability 1, *every* element of $\mathcal{P}_A$ gets assigned to one of the sets $F_1(A), G_1(A), F_2(A), G_2(A), \ldots$, so these sets form a partition of $\mathcal{P}_A$ as asserted.

Next we show that $\cup_{i \geq 1} F_i(A)$ is the set of accepted items in the target set A. A root represents an item that arrives before any overlapping items, so elements of $F_1(A)$ are accepted. Elements of $G_1(A)$ are rejected because they overlap with previously accepted 'parents' in $F_1(A)$. Elements of $F_2(A)$ are accepted because they have parents only in $G_1(A)$, and elements of $G_2(A)$ are rejected because they have accepted parents in $F_2(A)$. The argument proceeds in this way, formally by an induction which we leave to the reader. This procedure for determining which items have been accepted is essentially the same as that used by Caser and Hilhorst (1995), as the basis for a rigorous bound for λ (without determining its existence).

Finally we show that the sets $F_1, G_1, F_2, G_2, \ldots$ form a partition of $\mathcal{P}$. For $(X, T) \in \mathcal{P}$, let $C_{(X,T)}$ (the 'cluster at (X, T)') be the (random) set of ancestors of (X, T), that is, the set of $(Y, U) \in \mathcal{P}$ such that there is an oriented path from (Y, U) to (X, T). By Corollary 3.1, $C_{(X,T)}$ is finite for all $(X, T) \in \mathcal{P}$ with probability 1. Recursively define subsets $F_i(C_{(X,T)})$, and $G_i(C_{(X,T)})$, $i = 1, 2, \ldots$, in exactly the same manner as for $F_i(A), G_i(A)$, except that now we start out with the restriction of $\mathcal{G}$ to the (finite) set of vertices $C_{(X,T)}$, rather than to $\mathcal{P}_A$. In this case the algorithm will terminate at some point when there are no roots left. Since every finite subset of $C_{(X,T)}$ contains at least one root, the algorithm does not terminate until every element of $C_{(X,T)}$ has been assigned to one of the sets $F_i(C_{(X,T)})$, $G_i(C_{(X,T)})$, so these sets form a partition.

In the oriented graph $\mathcal{G}$, by definition there are no oriented edges of $\mathcal{G}$ from outside $C_{(X,T)}$ into $C_{(X,T)}$. For this reason, any root of the restriction of $\mathcal{G}$ to $C_{(X,T)}$ is also a root of the entire graph $\mathcal{G}$. Therefore $F_1(C_{(X,T)}) = F_1 \cap C_{(X,T)}$, and so $G_1(C_{(X,T)}) = G_1 \cap C_{(X,T)}$, and so on: for all j,

$$F_j(C_{(X,T)}) = F_j \cap C_{(X,T)} \quad \text{and} \quad G_j(C_{(X,T)}) = G_j \cap C_{(X,T)}.$$

Since $C_{(X,T)}$ is finite, (X,T) must lie in one of the sets $F_i(C_{(X,T)})$ or $G_i(C_{(X,T)})$, and therefore also in one of the sets F_i or G_i. Therefore these sets do indeed form a partition of $\mathcal{P}$ as asserted. ■

Define the indicator function $I(\cdot;A)$ defined for all points of $\mathcal{P}_A$, by $I(X,T;A) = 1$ if $(X,T) \in \cup_{i=1}^{\infty} F_i(A)$ and $I(X,T;A) = 0$ if $x \in \cup_{i=1}^{\infty} G_i(A)$. By Lemma 3.2, this function is well-defined, and takes the value 1 precisely on the set of locations in space-time of those items that are accepted in the the realization of the Palásti RSA model on the target set A, described at the start of this section. Therefore, for $B \subset \mathbb{R}^d$, the definition

$$N_t(B;A) = \sum_{j:X_j \in B, T_j \leq t} I(X_j, T_j; A).$$

is consistent with (3.1).

We now define the Palásti RSA process with target set $\mathbb{R}^d$, using the analogous partition of the whole of $\mathcal{P}$. For each point (X,T) of $\mathcal{P}$, Set $I(X,T) = 1$ if $(X,T) \in \cup_{i=1}^{\infty} F_i$ and $I(X,T) = 0$ if $(X,T) \in \cup_{i=1}^{\infty} G_i$. Then $I(X,T)$ is well-defined for all $(X,T) \in \mathcal{P}$, by Lemma 3.2. The following observation will be useful later on.

Remark 3.1 *Let (X,T) be a point of $\mathcal{P}$. The modification of $\mathcal{G}$ by addition or removal of edges with neither endpoint in $C_{(X,T)}$ has no effect on the value of $I(X,T)$ given by the algorithm just described.*

For bounded Borel $B \subset \mathbb{R}^d$, define the integer-valued random variables

$$N_t(B) = \sum_{\{j:X_j \in B, T_j \leq t\}} I(X_j, T_j); \qquad N(B) = \lim_{t \to \infty} N_t(B). \tag{3.5}$$

Clearly $N_t(B)$ is nondecreasing in t. Also, $N(B)$ is finite for bounded B because D has strictly positive volume, so there is a finite bound on the number of translates of D that can be packed into a bounded region.

Let ξ_t (respectively ξ) the point process associated with the counting measure $N_t(\cdot)$ (respectively $N(\cdot)$). That is, re-labelling the points of $\mathcal{P}$ in arbitrary order as $\{(X_j, T_j)\}_{j=1}^{\infty}$, let ξ_t and ξ be the random locally finite sets in $\mathbb{R}^d$ defined by

$$\xi_t = \{X_j : I(X_j, T_j) = 1, T_j \leq t\}; \quad \xi = \{X_j : I(X_j, T_j) = 1\}.$$

These are now rigorously defined in terms of the Poisson process $\mathcal{P}$ and the graphical construction. They are stationary point processes on $\mathbb{R}^d$.

4 Proof of the basic results

The proof of Theorem 2.1 uses the following multiparameter form of the Ergodic Theorem.

Lemma 4.3 *Suppose $(E, \mathcal{E}, P_0)$ is a probability space. Let $(\Omega, \mathcal{F}, P)$ be the probability space $\Omega := E^{\mathbb{Z}^d} = \prod_{z \in \mathbb{Z}^d} E$, equipped with the smallest σ-field such that each of the co-ordinate projections $X_z : \Omega \to E$, $z \in \mathbb{Z}^d$, is measurable, and with the probability measure P that makes the E-valued random variables $(X_z, z \in \mathbb{Z}^d)$ into independent identically distributed random elements of E with common probability distribution P_0.*

Suppose $(Y_z, z \in \mathbb{Z}^d)$ is a stationary random field on $(\Omega, \mathcal{F}, \mathcal{P}_0)$, meaning that $Y_{\mathbf{0}} : \Omega \to \mathbb{R}$ is a real-valued random variable on $(\Omega, \mathcal{F}, P)$, and for all $\omega \in \Omega$ and $z \in \mathbb{Z}^d$ we have $Y_z(\omega) = Y_{\mathbf{0}}(\tau_z(\omega))$, where the shift operator τ_z maps $\omega = (\omega_y, y \in \mathbb{Z}^d)$ to $\tau_z(\omega) = (\omega_{y+z}, y \in \mathbb{Z}^d)$. Suppose also that $\mathbb{E}\,|Y_{\mathbf{0}}| < \infty$.

Suppose $(A_n, n \geq 1)$ is a sequence of subsets of $\mathbb{Z}^d$ with $|\partial^{\mathbb{Z}} A_n|/|A_n| \to 0$. Then

$$\frac{\sum_{z \in A_n} Y_z}{|A_n|} \xrightarrow{L^1} \mathbb{E}\,Y_{\mathbf{0}} \qquad \text{as} \quad n \to \infty. \tag{4.6}$$

Proof. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^d$. Since P is a product measure, the shift τ_{e_1} is an ergodic transformation of Ω (see Petersen (1983), pp. 57-58). Given $\varepsilon > 0$, by the classical one-parameter Ergodic Theorem, we can choose $K > 0$ such that for all $n \geq K$, the average of $Y_{e_1}, Y_{2e_1}, \ldots, Y_{ne_1}$ is within an L^1 distance at most ε of $\mathbb{E}\,[Y_{\mathbf{0}}]$.

Divide A_n into one-dimensional *intervals* by which we mean maximal subsets of A_n of the form $(\mathbb{Z} \cap [a, b]) \times \{z_2\} \times \cdots \times \{z_d\}$, with $a, b, z_1, \ldots, z_d$ in $\mathbb{Z}$. Let A_n^* be the union of constituent intervals of length at least K. Let $k_n = |A_n|$ and $k'_n = |A_n^*|$. Since $|\partial A_n|/|A_n| \to 0$, $\lim_{n\to\infty}(k'_n/k_n) = 1$. Writing $\|X\|_1$ for the L^1-norm (expected absolute value) of any random variable X, we have

$$\begin{aligned} \left\| (k_n^{-1} \sum_{z \in A_n} Y_z) - \mathbb{E}\,[Y_{\mathbf{0}}] \right\|_1 &\leq k_n^{-1} \left\| (\sum_{z \in A_n^*} Y_z) - k'_n \mathbb{E}\,[Y_{\mathbf{0}}] \right\|_1 \\ &+ k_n^{-1} \left\| \left(\sum_{z \in A_n \setminus A_n^*} Y_z \right) - (k_n - k'_n)\mathbb{E}\,[Y_{\mathbf{0}}] \right\|_1 . \end{aligned} \tag{4.7}$$

By the choice of K and translation-invariance, for each interval I of length at least K the average of $Y_z, z \in I$, is within an L^1-distance ε of $\mathbb{E}\,[Y_{\mathbf{0}}]$. Therefore the first term on the right hand side of (4.7) is at most ε, while the second term tends to zero because $(k'_n/k_n) \to 1$. Therefore the left side of (4.7) is less than 2ε for large n, and (4.6) follows. ■

For each $z \in \mathbb{Z}^d$, let $\mathcal{N}_z$ denote the set of $z' \in \mathbb{Z}^d$ such that $\|z' - z\| \leq 1$ (including z itself). Let Q_z^+ denote the enlarged cube $\cup_{z' \in \mathcal{N}_z} Q_z$, and define the 'jamming time'

$$J_z := \inf\{t \geq 0 : N_t(Q_z^+) = N(Q_z^+)\}.$$

The value of J_z is clearly finite. At time J_z, with probability 1 the cube Q_z is jammed, meaning that the set V of $x \in Q_z$ such that $x + D$ does not intersect any of the accepted sets $X_i + D$ with $T_i \leq J_z$ and $I(X_i, T_i) = 1$ has zero Lebesgue measure. Indeed, if its measure were not zero, then at some later time T after J_z there would be a Poisson arrival in the set V, and this item would either be accepted itself, contradicting the definition of J_z, or be blocked by some accepted item arriving in Q_z^+ between times J_z and T, also contradicting the definition of J_z.

Given $A \subset \mathbb{R}^d$, let $\mathcal{L}_A$ be the lattice discretization of A, by which we mean the set of $z \in \mathbb{Z}^d$ such that $Q_z \cap A \neq \emptyset$. Given also $K > 0$, let $\mathcal{L}^o_{A,K}$ be the lattice K-interior of A defined by

$$\mathcal{L}^o_{A,K} := \{z \in \mathcal{L}_A : B_K(z) \subseteq A\},$$

and let $\mathcal{L}^{\mathrm{b}}_{A,K} = \mathcal{L}_A \setminus \mathcal{L}^o_{A,K}$ (a set of 'boundary' lattice points).

Recall from (3.3) that for $y, z \in \mathbb{Z}^d$, $E_t(y,z)$ is the event that y affects z before time t. Given $K > 0$, define the event $F_{t,K}(z)$ by

$$F_{t,K}(z) = \cup_{z' \in \mathcal{N}_z} \cup_{y \in \mathbb{Z}^d, \|y-z\| \geq K} E_t(y, z'). \tag{4.8}$$

Lemma 4.4 *Suppose $A \subset \mathbb{R}^d$, suppose $K > 0$ is an integer, suppose $t > 0$, and suppose $z \in \mathcal{L}^o_{A,K}$. Then we have the following event inclusions (ignoring events of probability zero):*

$$\{N_t(Q_z; A) \neq N_t(Q_z\} \subset F_{t,K}(z) \tag{4.9}$$

and

$$\{N_t(Q_z; A) \neq N(Q_z; A)\} \subset F_{t,K}(z) \cup \{J_z > t\}. \tag{4.10}$$

Proof. Suppose that $F_t(z)$ does *not* occur, i.e. there does not exist $y \in \mathbb{Z}^d$ with $\|y - z\| \geq K$, and $z' \in \mathcal{N}_z$, such that y affects z' before time t.

Then for all Poisson points $(X, T) \in Q_z^+ \times [0, t]$, the cluster $C_{(X,T)}$ is contained in $B_K(z) \times [0, t]$, and therefore in $A \times [0, t]$. Therefore $I(X, T) = I(X, T; A)$ for all such (X, T), so $N_t(Q_z; A) = N(Q_z)$, and (4.9) follows.

If also $J_z \leq t$, the set $\{X_i \in Q_z^+ : I(X_i, T_i) = 1, T_i \leq t\}$ (of points accepted in Q_z^+ before time t) jams Q_z, so no arrivals in Q_z after time t are accepted and hence $N_t(Q_z; A) = N(Q_z; A)$. This gives us (4.10). ■

In the proof of Theorem 2.1, we shall repeatedly use the fact that the packing constant c_1 defined by

$$c_1 := \max\{k : \exists \{x_1, \ldots, x_k\} \subset Q_{\mathbf{0}} \text{ with } x_1 + D, \ldots, x_k + D \text{ disjoint}\} \tag{4.11}$$

is finite, because D has positive volume and is bounded.

Lemma 4.5 *Suppose $(A_n)_{n \geq 1}$ is a sequence of bounded nonnull Borel subsets of $\mathbb{R}^d$ satisfying $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$. Then as $n \to \infty$,*

$$\frac{N(A_n)}{|A_n|} \xrightarrow{L^1} \lambda; \quad \frac{N_t(A_n)}{|A_n|} \xrightarrow{L^1} \lambda_t, \tag{4.12}$$

where we set $\lambda = \mathbb{E}\, N(Q_{\mathbf{0}})$ and $\lambda_t = \mathbb{E}\, N_t(Q_{\mathbf{0}})$.

Proof. Since $|A_n| \leq |\mathcal{L}_{A_n}| \leq |A_n| + |\partial_1 A_n|$, and since $|\mathcal{L}^{\mathrm{b}}_{A_n,K}| \leq |\partial_{K+1} A_n|$, we have as $n \to \infty$ that

$$\frac{|\mathcal{L}_{A_n}|}{|A_n|} \to 1; \quad \frac{|\mathcal{L}^{\mathrm{b}}_{A_n,K}|}{|A_n|} \to 0. \tag{4.13}$$

We wish to apply the Ergodic Theorem (Lemma 4.3). Let $(\mathcal{P}_z, z \in \mathbb{Z}^d)$ be a family of independent homogeneous Poisson processes on process of rate 1 on on $Q_{\mathbf{0}} \times [0,\infty)$. More formally, let $(E, \mathcal{E}, P_0)$ be the space of point configurations on $Q_{\mathbf{0}} \times [0,\infty)$, equipped with the probability measure corresponding to a homogeneous Poisson point process of rate 1 on $Q_{\mathbf{0}} \times [0,\infty)$, and let $(\Omega, \mathcal{F}, P)$ be the probability space $\Omega := \prod_{x \in \mathbb{Z}^d} E$ equipped with the smallest σ-field such that each of the co-ordinate projections $\mathcal{P}_z : \Omega \to E$, $z \in \mathbb{Z}^d$, is measurable, and with the probability measure P that makes the E-valued random variables $(\mathcal{P}_z, z \in \mathbb{Z}^d)$ into independent identically distributed random elements of E with common distribution P_0.

Assume the Poisson point process $\mathcal{P}$ on $\mathbb{R}^d \times [0,\infty)$, used as the basis of the graphical construction, is generated by taking the union of the independent Poisson point processes $\sigma_z(\mathcal{P}_z)$, where $\sigma_z(\mathcal{P}_z)$ is the image of the point set $\mathcal{P}_z$ under the translation σ_z which sends each point $(x,t) \in Q_{\mathbf{0}} \times [0,\infty)$ to $(x+z,t)$. For $z \in \mathbb{Z}^d$, define $Y_z = N(Q_z)$. Since the graph $\mathcal{G}$ is translation-invariant, in the sense that the relation $(X,T) \to (X',T')$ is invariant under the translation σ_z of both (X,T) and (X,T'), the variables $(Y_z, z \in \mathbb{Z}^d)$ form a stationary random field in the sense of Lemma 4.3. By that result,

$$\frac{N\left(\cup_{z \in \mathcal{L}^o_{A_n,1}} Q_z\right)}{|\mathcal{L}^o_{A_n,1}|} = \frac{\sum_{z \in \mathcal{L}^o_{A_n,1}} Y_z}{|\mathcal{L}^o_{A_n,1}|} \xrightarrow{L^1} \lambda. \tag{4.14}$$

Suppose $A \subset \mathbb{R}^d$. Since $A \setminus \cup_{z \in \mathcal{L}^o_{A,1}} Q_z$ is contained in the union of at most $|\mathcal{L}^{\mathrm{b}}_{A,1}|$ unit cubes, the definition (4.11) of the packing constant c_1 yields the uniform bound

$$0 \le N(A) - N\left(\cup_{z \in \mathcal{L}^o_{A,1}} Q_z\right) \le c_1 |\mathcal{L}^{\mathrm{b}}_{A,1}|,$$

so by (4.13) and (4.14) we obtain the first convergence result in (4.12). The proof of the second convergence result is entirely analogous.

Lemma 4.6 *Suppose $(A_n)_{n\ge1}$ is a sequence of bounded nonnull Borel subsets of $\mathbb{R}^d$ satisfying $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$. Then for any $t \in (0,\infty)$,*

$$\frac{|N_t(A_n) - N_t(A_n; A_n)|}{|A_n|} \xrightarrow{L^1} 0 \qquad \text{as} \quad n \to \infty.$$

Proof. Choose integer $K > 1$. Since $|N_t(Q_z; A) - N_t(Q_z \cap A)|$ is uniformly bounded by c_1 for all $z \in \mathbb{Z}^d$, using Lemma 4.4 we obtain

$$\begin{aligned} |A_n|^{-1} |N_t(A_n; A_n) - N_t(A_n)| &\le |A_n|^{-1} \sum_{z \in \mathcal{L}_{A_n}} |N_t(Q_z; A_n) - N(Q_z \cap A_n)| \\ &\le \frac{c_1 |\mathcal{L}^{\mathrm{b}}_{A_n,K}|}{|A_n|} + \frac{c_1 \sum_{z \in \mathcal{L}^o_{A_n,K}} \mathbf{1}_{F_{t,K}(z)}}{|A_n|}. \end{aligned}$$

In the the right hand side the first term tends to zero by (4.13) while the second term converges in L^1 to $c_1 P[F_{t,K}(\mathbf{0})]$ by the Ergodic Theorem (cf. the proof of (4.14)). Taking $K \to \infty$ and using Lemma 3.1 we obtain the result. ∎

Lemma 4.7 *Suppose $(A_n)_{n\ge1}$ is a sequence of bounded nonnull Borel subsets of $\mathbb{R}^d$ satisfying $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$. Then for any $t \in (0,\infty)$,*

$$\limsup_{n\to\infty} \mathbb{E}\left[\frac{|N_t(A_n; A_n) - N(A_n; A_n)|}{|A_n|}\right] \le P[J_{\mathbf{0}} > t].$$

Proof. Let $K > 1$. By Lemma 4.4,

$$\frac{|N_t(A_n;A_n) - N(A_n;A_n)|}{|A_n|} \leq \frac{\sum_{z \in \mathcal{L}_{A_n}} |N_t(Q_z;A_n) - N(Q_z \cap A_n;A_n)|}{|A_n|}$$
$$\leq \frac{c_1 |\mathcal{L}^{\mathrm{b}}_{A_n,K}|}{|A_n|} + \frac{c_1 \sum_{z \in \mathcal{L}^o_{A_n,K}} \mathbf{1}_{F_{t,K}(z) \cup \{J_z > t\}}}{|A_n|}.$$

In the the right hand side the mean of the first term tends to zero by (4.13), while the mean of the second term tends to $P[F_{t,K}(\mathbf{0}) \cup \{J_{\mathbf{0}} > t\}]$ by stationarity. Taking $K \to \infty$ and using Lemma 3.1 gives us the result. ∎

Proof of Theorem 2.1. Let $t > 0$. Since $|N(A) - N(A;A)|$ is bounded by

$$|N(A) - N_t(A)| + |N_t(A) - N_t(A;A)| + |N_t(A;A) - N(A;A)|,$$

by combining Lemmas 4.5, 4.6, and 4.7, we obtain

$$\limsup_{n\to\infty} \mathbb{E}\left[\frac{|N(A_n) - N(A_n;A_n)|}{|A_n|}\right] \leq \lambda - \lambda_t + P[J_{\mathbf{0}} > t],$$

and taking $t \to \infty$, this gives us the result. ∎

Almost sure convergence. One may view Theorem 2.1 as a weak law of large numbers, and ask if there is an associated strong law, with convergence in pth moment replaced by almost sure convergence. The question makes sense now that we are assuming that the Palásti models on different target sets A are all defined in terms of a single Poisson process $\mathcal{P}$ in space-time.

In Theorem 2.1 as stated, the only requirement on the sets A_n is a very mild condition of negligible boundary, but to obtain almost sure convergence we need a more restricted class of sets A_n. This is because we need a form of multiparameter Ergodic Theorem with almost sure convergence. A sufficient condition would be to make the extra assumptions that (i) the sets A_n are all 'rectangles', i.e. products of intervals, and (ii) $\liminf(A_n) = \mathbb{R}^d$. In that case, sets of the form $\mathcal{L}^o_{A_n}$ will be lattice rectangles satisfying $\liminf(\mathcal{L}_{A_n}) = \mathbb{Z}^d$, and by a multiparameter Ergodic Theorem (e.g. Dunford and Schwartz (1958), theorem VIII.6.9) the L^1 convergence in the Ergodic theorem (Lemma 4.3) can be replaced by almost sure convergence, and therefore in Lemmas 4.5 and 4.6, the L^1 convergence can be replaced by a.s. convergence, while Lemma 4.7 can be changed to an a.s. bound on the limsup. Therefore, with the extra assumptions (i) and (ii) above, the proof of Theorem 2.1 can indeed be modified to give almost sure convergence.

Proof of Theorem 2.2. Let ξ be the point process associated with the counting measure $N(\cdot)$, and let ξ_t be the point process associated with the counting measure $N_t(\cdot)$. Suppose $(A_n)_{n\geq 1}$ is a sequence of target sets with $\liminf(A_n) = \mathbb{R}^d$. We shall show that ξ^{A_n} converges weakly to ξ, and for all $t > 0$, $\xi_t^{A_n}$ converges weakly to ξ_t.

With $N_t(B;A)$ and $N_t(B)$ all defined in terms of a single Poisson process $\mathcal{P}$ using the graphical representation in Section 3, it suffices to prove that for any bounded Borel $B \subset \mathbb{R}^d$ we have *almost sure* convergence

$$N_t(B;A_n) \to N_t(B); \qquad N(B;A_n) \to N(B). \tag{4.15}$$

There is no loss of generality in assuming that B is contained in some unit cube Q_z (since if not we simply sum the contributions of intersections of B with a finite collection of such cubes), and from now on we do so.

Let $t > 0$. Recall the definition of events $E_t(y, z)$ and $F_{t,K}(z)$ at (3.3) and (4.8). Define the random variable

$$M(t) := \min\{K \in \mathbb{N} : F_{t,K}(z) \text{ does not occur}\}$$

which is almost surely finite by Corollary 3.1. Then $F_{t,M(t)}(z)$ does not occur, so if n is sufficiently large so that $z \in \mathcal{L}^o_{A_n,M(t)}$, then by the proof of (4.9) in Lemma 4.4, $N_t(B; A) = N_t(B)$. This proves the first part of (4.15).

Let $T = J_z$, which is random but almost surely finite. Then neither $F_{T,M(T)}(z)$ nor $\{J_z > T\}$ occurs, so if n is sufficiently large so that $z \in \mathcal{L}^o_{A_n,M(T)}$, then by the proof of (4.10) in Lemma 4.4,

$$N(B; A) = N_T(B; A) = N_T(B) = N(B),$$

the last inequality coming from the definition of $T = J_z$ as the jamming time. This gives us the second part of (4.15). ■

5 Non-uniform distributions

This section is concerned with the following model. Suppose $(X_1, X_2, X_3, \dots)$ are independent identically distributed by random d-vectors with common probability density function $f : \mathbb{R}^d \to [0, \infty)$. Let D be a specified bounded open set in $\mathbb{R}^d$ with $\mathbf{0} \in D$, and for $a > 0$ let aD denote the set $\{ax : x \in D\}$. For each positive integer n, an n-model generating a random variable denoted N^*_n is defined as follows. For $i = 1, 2, \dots$, item i is accepted in the n-model if $(X_i + n^{-1/d}D) \cap (X_j + n^{-1/d}D) = \emptyset$ for all previous items $j, j < i$ that were accepted in the n-model. For $B \subseteq \mathbb{R}^d$, let $N^*_n(B)$ be the number of items that arrive in B and are accepted for the n-model, i.e. the number of i for which $X_i \in B$ and item i is accepted. Write simply N^*_n for $N^*_n(\mathbb{R}^d)$, the total number of items accepted.

For the special case where f is a uniform distribution over the unit cube, (or some other fixed set with a reasonably well-behaved boundary), Theorem 2.1 and a simple spatial rescaling argument immediately give convergence of N^*_n/n in p-th moment to λ (the same limit as in Theorem 2.1). In the present section we seek extensions to non-uniform density functions f. In one dimension only, non-uniformity was incorporated into extensions of Rényi's result by Ney (1962); see also the remarks in Section 6.2 of Dvoretzky and Robbins (1964). The non-uniform distribution could represent RSA of particles onto a surface with variable adhesivity, or sequential arrival of members of a territorial biological species in a previously unoccupied area of land where some parts of the terrain are more friendly than others. This form of non-homogeneous SSI has been used in statistical simulations by Pretzsch (1995), described in English by Stoyan and Stoyan (1995).

Recall that the *support* of the function f is the intersection of all closed sets $F \subseteq \mathbb{R}^d$ such that $\int_{\mathbb{R}^d \setminus F} f(x)dx = 0$.

Theorem 5.1 *Suppose f is Riemann integrable and $\{x \in \mathbb{R}^d : f(x) > 0\}$ is bounded and Riemann measurable with measure $|S|$. Let $\lambda = \lambda(d, D)$ be the constant appearing in Theorem 2.1. Then in the above model, for all $p \in [1, \infty)$,*

$$n^{-1} N_n^* \xrightarrow{L^p} \lambda |S|. \tag{5.1}$$

We emphasize that given d and D, the limit in (5.1) depends on f only via the Lebesgue measure $|S|$ of its support. Estimates of λ by simulation studies for $d \geq 2$ in the uniform case, and also the exact value of λ given by (1.1) for $d = 1$ in the uniform case, are therefore also applicable via (5.1) in the non-uniform case under consideration here.

Proof. Let $\varepsilon > 0$. Define dyadic step functions f_n^+ and f_n^- approximating f from above and below, as follows. Divide $\mathbb{R}^d$ into half-open n-dyadic cubes, that is, sets of the form $\prod_{j=1}^d (a_j 2^{-n}, (a_j + 1)2^{-n}]$ with all $a_j \in \mathbb{Z}$. Given $x \in \mathbb{R}^d$, if $Q'_n(x)$ is the n-dyadic cube containing x, let $f_n^+(x) = \sup\{f(y) : y \in Q'_n(x)\}$, and $f_n^-(x) = \inf\{f(y) : y \in Q'_n(x)\}$. Let S_n^+ be the support of f_n^+ and let $S_n^- := \{x \in \mathbb{R}^d : f_n^-(x) > 0\}$. Then $S_n^- \subseteq S \subseteq S_n^+$. Since $\{x : f(x) > 0\}$ is assumed Riemann measurable, it is possible to choose $n_0 = n_0(\varepsilon)$ such that

$$|S_{n_0}^+ \setminus S_{n_0}^-| < \varepsilon. \tag{5.2}$$

The restriction of $f_{n_0}^-$ to $S_{n_0}^-$ is strictly positive and takes finitely many values, so there is a constant $\delta_2 = \delta_2(\varepsilon)$ such that $f_{n_0}^-(x) \geq \delta_2, x \in S_{n_0}^-$.

Let $\varepsilon > 0$. Using the Riemann integrability of f, choose $n_1 = n_1(\varepsilon) \geq n_0$ such that

$$\int_{\mathbb{R}^d} (f_{n_1}^+(x) - f_{n_1}^-(x))dx \leq \varepsilon^2 \delta_2,$$

which implies by Markov's inequality that

$$|\{x \in \mathbb{R}^d : f_{n_1}^+(x) > f_{n_1}^-(x) + \varepsilon\delta_2\}| \leq \varepsilon. \tag{5.3}$$

Define the 'well-behaved' set

$$\Gamma_1 := \{x \in S_{n_0}^- : f_{n_1}^+(x) \leq f_{n_1}^-(x) + \varepsilon\delta_2\}, \tag{5.4}$$

and the 'exceptional' set

$$\Gamma_2 := \left(S_{n_0}^+ \setminus S_{n_0}^-\right) \cup \{x \in S_{n_0}^- : f_{n_1}^+(x) > f_{n_1}^-(x) + \varepsilon\delta_2\}.$$

Then

$$\Gamma_1 \subset S \subset (\Gamma_1 \cup \Gamma_2). \tag{5.5}$$

By (5.2) and (5.3), the set Γ_2 is a union of n_0-dyadic and n_1-dyadic cubes with total volume

$$|\Gamma_2| < 2\varepsilon. \tag{5.6}$$

By dividing each of these cubes into subcubes of volume $1/n$, for large enough n we have the uniform deterministic bound

$$N_n^*(\Gamma_2) \leq 3nc_1\varepsilon, \tag{5.7}$$

with the packing constant c_1 defined at (4.11).

Choose one of the n_1-dyadic cubes constituting Γ_1, and let it be denoted A. Let x_0 denote the centre of A. Let g_1 be the (constant) value taken by the function $f_{n_1}^-$ on the set A. Note that $g_1 \geq \delta_2$ and by (5.4),

$$g_1 \leq f(x) \leq g_1(1+\varepsilon), \qquad x \in A. \tag{5.8}$$

The next aim is to estimate $N_n^*(A)$. On a suitable probability space, let $\mathcal{P}$ be a homogeneous Poisson process of rate 1 on $\mathbb{R}^d \times [0,\infty)$. Consider the Palásti RSA model, with exclusion based on overlap of translates of D (rather than $n^{-1/d}D$) and with arrivals in space-time given by the points of $\mathcal{P}$. For $B \subset \mathbb{R}^d$, let $N(B)$ be the number of items accepted in $B \times [0,\infty)$, i.e. let $N(B)$ be precisely as same as defined at (3.5).

Let A_n denote the cube of side $n^{1/d}2^{-n_1}$ centred at the origin; note that

$$|A_n| = n|A|. \tag{5.9}$$

Let $\mathcal{P}_{1,n}$ be the restriction of $\mathcal{P}$ to A_n, and assume that on the same probability space, there is an independent non-homogeneous Poisson process $\mathcal{P}_{2,n}$ on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $h_n(y)dydt$ ($y \in \mathbb{R}^d, t > 0$), where we set

$$h_n(y) := g_1^{-1}(f(x_0 + n^{-1/d}y) - g_1), \qquad y \in A_n;$$
$$h_n(y) := g_1^{-1}f(x_0 + n^{-1/d}y), \qquad y \in \mathbb{R}^d \setminus A_n.$$

Later on we shall use the fact that this definition and (5.8) together imply that

$$0 \leq h_n(y) \leq \varepsilon, \qquad y \in A_n. \tag{5.10}$$

The union of $\mathcal{P}_{1,n}$ and $\mathcal{P}_{2,n}$ is a non-homogeneous Poisson process on $\mathbb{R}^d \times [0,\infty)$ with intensity measure

$$g_1^{-1}f(x_0 + n^{-1/d}y)dydt.$$

Consider a RSA model in which items arrive in $\mathbb{R}^d$ at points in space-time given by the points of the Poisson process $\mathcal{P}_{1,n} \cup \mathcal{P}_{2,n}$, and exclusion is based on overlaps of translates of D, as in the case of $N(B)$ above, and in the basic RSA model described in Section 3. Let $N_n^{**}(A_n)$ denote the total number of items accepted with arrivals in $A_n \times [0,\infty)$.

Define the linear transformation $\tau_n : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d \times [0,\infty)$ which maps (y,t) to $(x_0 + n^{-1/d}y, t)$ (so that the image of $A_n \times [0,\infty)$ under τ_n is $A \times [0,\infty)$). By the transformation theory of Poisson processes, as given for example in Resnick (1987, proposition 3.7), the image of the Poisson process $\mathcal{P}_{1,n} \cup \mathcal{P}_{2,n}$ under the mapping τ_n is a nonhomogeneous Poisson process on $\mathbb{R}^d \times [0,\infty)$ whose intensity measure is $(n/g_1)f(x)dxdt$; in other words, a sequence of independent variables on $\mathbb{R}^d$ with common probability density function f, arriving at the arrival times of a homogeneous Poisson process in $[0,\infty)$ of rate n/g_1. Let us assume that the input for the n-model is given by the positions in $\mathbb{R}^d$ of the arrivals of this transformed Poisson process, taken in time-order. In that case, since the transformation τ shrinks sets by a factor of $n^{-1/d}$ and maps points in A_n to points in A, we have the identity

$$N_n^{**}(A_n) = N_n^*(A). \tag{5.11}$$

Next, we aim to approximate $N_n^{**}(A_n)$ by $N(A_n)$. Let $t > 0$ and let $K > 0$ with $(K-2)\delta_1 > t$. For $z \in \mathbb{Z}^d$, let U_z be the first arrival time of the Poisson process $\mathcal{P}_{2,n}$ in $B_K(z)$, i.e., labelling the points of $\mathcal{P}_{2,n}$ as $\{(Y_i,T_i)\}_{i=1}^{\infty}$,

$$U_z = \min\{T_i : Y_i \in B_K(z)\}.$$

Since at most c_1 items are packed into any unit cube, using notation from Section 4 we have

$$\begin{aligned}\mathbb{E}\,|N_n^{**}(A_n) - N(A_n)| &\le \sum_{z \in \mathcal{L}_{A_n}} \mathbb{E}\,|N_n^{**}(Q_z \cap A_n) - N(Q_z \cap A_n)| \\ &\le c_1 \sum_{z \in \mathcal{L}^o_{A_n,K}} P[N_n^{**}(Q_z) \neq N(Q_z)|; J_z \le t; U_z > t] \\ &\quad + c_1 \sum_{z \in \mathcal{L}^o_{A_n,K}} (P[J_z > t] + P[U_z \le t]) + c_1 |\mathcal{L}^{\mathrm{b}}_{A_n,K}|. \end{aligned} \tag{5.12}$$

For $z \in \mathcal{L}^o_{A_n,K}$ the set $B_K(z)$ is contained in A_n, and therefore by (5.10), the expected total number of points of $\mathcal{P}_{2,n}$ in $B_K(z) \times [0,t]$ is bounded by $(2K)^d t\varepsilon$, so that

$$P[U_z \le t] \le (2K)^d t\varepsilon, \qquad z \in \mathcal{L}^o_{A_n,K}. \tag{5.13}$$

Turning to the first sum in the right hand side of (5.12), for $z \in \mathcal{L}^o_{A_n,K}$ we assert that

$$P[N_n^{**}(Q_z) \neq N(Q_z); J_z \le t; U_z > t] \le P[F_{t,K-1}(z)] \tag{5.14}$$

$$\le |\mathcal{N}_z| P \cup_{\|z\| > K-2} E_t(z, \mathbf{0})] \le 2(3^{d+2-K}). \tag{5.15}$$

The first inequality (5.14) is obtained by a similar argument to Lemma 4.4; if $F_{t,K-1}(z)$ does not occur, then for $(X,T) \in Q_z^+ \times [0,t]$, the cluster $C_{(X,T)}$ is contained in the cylinder $B_{K-1}(z) \times [0,t]$. If also $U_z > t$, then there are no points of $\mathcal{P}_{n,2}$ in the slightly larger cylinder $B_K(z) \times [0,t]$, and so none of these points is attached to $C_{(X,T)}$ in the graph. Therefore the set of accepted points in Q_z^+ up to time t is the same for the Poisson process $\mathcal{P}_{1,n} \cup \mathcal{P}_{2,n}$ as it is for $\mathcal{P}_{1,n}$, and if also $J_z \le t$, then no point is accepted in Q_z after time t because it is jammed. This proves the inequality (5.14). The last inequality in (5.15) comes from Lemma 3.1 and the fact that we assumed $(K-2)\delta_1 > t$. By (5.15) and (5.13), the left hand side of (5.12) is bounded by

$$c_1 |\mathcal{L}^o_{A_n,K}| (2(3^{d+2-K}) + P[J_{\mathbf{0}} > t] + (2K)^d t\varepsilon) + c_1 |\mathcal{L}^{\mathrm{b}}_{A_n,K}|. \tag{5.16}$$

By the Ergodic Theorem (Lemma 4.5), we have $(N(A_n)/|A_n|) \xrightarrow{L^1} \lambda$, and therefore by (5.9), (5.11) and the bound (5.16) for (5.12),

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left|\frac{N_n^*(A)}{n} - \lambda|A|\right| &= |A| \limsup_{n\to\infty} \mathbb{E}\left|\frac{N_n^{**}(A_n)}{|A_n|} - \lambda\right| \\ \le |A| \limsup_{n\to\infty} &\left(\mathbb{E}\left|\frac{N_n^{**}(A) - N(A_n)}{|A_n|}\right| + \mathbb{E}\left|\frac{N(A_n)}{|A_n|} - \lambda\right|\right) \\ &\le c_1 |A| \left(2 \cdot 3^{d+2-K} + P[J_{\mathbf{0}} > t] + \varepsilon(2K)^d t\right). \end{aligned}$$

Summing over the constituent cubes A of Γ_1 (a finite collection of cubes), and also using (5.5), (5.6) and (5.7), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left|\frac{N_n^*}{n} - \lambda|S|\right| &\le \limsup_{n\to\infty} \mathbb{E}\left|\frac{N_n^*(\Gamma_1)}{n} - \lambda|\Gamma_1|\right| \\ &\quad + \limsup_{n\to\infty} \mathbb{E}\left|\frac{N_n^*(S \setminus \Gamma_1)}{n} - \lambda|S \setminus \Gamma_1|\right| \\ \le c_1 |\Gamma_1| &\left(2 \cdot 3^{d+2-K} + P[J_{\mathbf{0}} > t] + \varepsilon(2K)^d t\right) + \varepsilon(3c_1 + 2). \end{aligned}$$

Letting $\varepsilon \to 0$, then $K \to \infty$, and then $t \to \infty$, we obtain $(N_n^*/n) \xrightarrow{L^1} \lambda|S|$, and convergence in p-th moment for $p > 1$ follows because $N_n^*/|A_n|$ is uniformly bounded by a constant. ■

6 Random shapes, random types

In the basic Palásti model of RSA, each incoming item is a translate of the same d-dimensional set D. It is natural to ask what happens if this condition is relaxed by allowing for several kinds of set. In the case $d = 1$, variable length 'cars' are incorporated into the models of Ney (1962) and Mannion (1979). In the case $d = 2$, the incoming items could be disks of variable radius, squares of variable orientation, rectangles of variable aspect ratio, and so on. Many of the recent simulation studies have been concerned with these types of RSA model. See the surveys mentioned in the introduction, for example sections 6.2 and 7 of Bartelt and Privman (1991).

More generally, one may have several types of item, with arbitrary exclusion rules between items of different types at different locations, not necessarily induced by each type representing a particular size or shape. For example, Itoh and Shepp (1999) have recently considered a one-dimensional parking model where there are two types, which they call 'spins'; items of the same type exclude each other up to range a and items of different type exclude each other up to range 1. They assert that for this model the mean number $f(x)$ of items accepted of a particular type in an interval of length x is subadditive in x, but Shepp (personal communication) has subsequently observed that this is not really true, so some other argument is needed to show convergence of $f(x)/x$ as $x \to \infty$. The approach here provides a method to prove this, as well as higher dimensional analogues.

We extend Theorem 2.1 by allowing for items which are of *random type*. The type can represent the shape of an item, or its 'spin', or any other property of the item. Let $(\mathbb{F}, \mathcal{F})$ be an arbitrary measurable space (the space of possible types), and let P_1 be a probability measure on $(\mathbb{F}, \mathcal{F})$, representing the distribution of types of random inputs. Suppose

$$\Phi : \mathbb{R}^d \times \mathbb{F} \times \mathbb{F} \to \{0, 1\}$$

is a measurable function (with respect to product measure), the so-called 'exclusion function', where $\Phi(x, \Xi, \Xi')$ takes the value 0 if an item of type Ξ excludes any subsequently arriving item of type Ξ' with relative displacement x. For example, in the case of spatial exclusion by items with random shape, one could take $\mathbb{F}$ to be a space of subsets of $\mathbb{R}^d$ and set $\Phi(x, \Xi, \Xi')$ to be 0 only if $\Xi \cap (x + \Xi') = \emptyset$. A general theory of random subsets of $\mathbb{R}^d$ can be found in Matheron (1975) and in condensed form in Stoyan, Kendall and Mecke (1995). In particular, results and remarks of Matheron (1975), pages 28, 9, and 13, imply that the function Φ just defined really is measurable.

Assume that the exclusion function has a positive 'hard core' component and a finite range. In other words, assume that that there are constants $0 < r_0 < r_1$ such that for $(P_1 \times P_1)$-almost all pairs (Ξ, Ξ'), we have

$$\Phi(x; \Xi, \Xi') = 0, \qquad \|x\| \leq r_0$$

and

$$\Phi(x; \Xi, \Xi') = 1, \qquad \|x\| \geq r_1.$$

The model goes as follows. Given a bounded nonnull Borel target set $A \subset \mathbb{R}^d$, let $X_1, X_2, \ldots$ be independent d-vectors uniformly distributed over A, and let $\Xi_1, \Xi_2, \ldots$ be independent random elements of $\mathbb{F}$ with common probability distribution P_1. We refer to X_i as the *location* of the i-th incoming item, and to Ξ_i as its *type*. Recursively, let the

i-th item be accepted if $\Phi(X_i - X_j, \Xi_j, \Xi_i) = 1$ for all $j < i$ such that j was accepted; otherwise the i-th item is rejected.

As in earlier results, it is convenient to assume Poisson input; in this case, let us take $\mathcal{P}$ to be a Poisson process on $\mathbb{R}^d \times [0, \infty) \times \mathbb{F}$ with intensity given by the product of Lebesgue measure on $\mathbb{R}^d \times [0, \infty)$ with the probability measure P_1 on $\mathbb{F}$. Each point of this Poisson process is a triple (X, T, Ξ), with $X \in \mathbb{R}^d$ representing location of an incoming item, $T \in [0, \infty)$ representing its time of arrival, and $\Xi \in \mathbb{F}$ its type. Then arriving items have independent identically distributed types with distribution P_1, and it is assumed that the input to the model as described in the preceding paragraph comes from the locations and types of the points of this Poisson process which lie in $A \times [0, \infty) \times \mathbb{F}$, taken in time-order.

Given $B \subset \mathbb{R}^d$, let $N(B; A)$ be the total number of items accepted whose location is in B. This is finite because only a finite number of translates of $[-r_0, r_0]^d$ can be packed into A. Also of interest are other counts concerned with accepted sets. For example, one might be interested in the total volume of accepted sets in B (which is trivially deduced from $N(B; A)$ in the basic Palásti model when P_1 is degenerate), or in the number of accepted sets of a particular kind. With this in mind, for any bounded measurable function $h : \mathbb{F} \to \mathbb{R}$, let us define $N^h(B; A)$ be the sum of the value of $h(\Xi_i)$ over all accepted items in B, or in other words,

$$N_t^h(B; A) = \sum_{i=1}^{\infty} h(\Xi_i) \mathbf{1}\{i \text{ accepted}; X_i \in B, T_i \leq t\},$$

and $N^h(B; A) = \lim_{t \to \infty} N_t^h(B; A)$.

In the case where the function h is the function $\mathbf{1}$ that is identically 1, we write simply $N(B; A)$ for $N^{\mathbf{1}}(B; A)$ the total number of accepted items in B, and and $N_t(B; A)$ for $N_t^{\mathbf{1}}(B; A)$, the total number of accepted items up to time t. Let ξ_t^A and ξ^A be the point processes associated with the counting measures $N_t(\cdot; A)$ and $N(\cdot; A)$.

Theorem 6.1 *There is a constant $\lambda > 0$, depending on the dimension d, the choice of measure P_1, and the function h, such that if $(A_n)_{n \geq 1}$ is a sequence of nonnull bounded Borel subsets of $\mathbb{R}^d$ satisfying $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$, then for any $p \in [1, \infty)$,*

$$\frac{N^h(A_n; A_n)}{|A_n|} \xrightarrow{L^p} \lambda. \tag{6.1}$$

Moreover, there exist point processes ξ and $\xi_t, t > 0$, such that if $(A_n)_{n \geq 1}$ is any sequence of target sets with $\liminf(A_n) = \mathbb{R}^d$, the sequence of point processes ξ^{A_n} converges weakly to ξ, and for all $t > 0$, the sequence of point processes $\xi_t^{A_n}$ converges weakly to ξ_t.

In this model, the distribution of types of accepted items will typically differ from the distribution P_1 of types of incoming items. For example, for balls of varying sizes, the distribution of accepted items will be biased towards smaller items that can fit more easily into gaps between previously accepted items. It might also be of interest to consider a model without this feature, namely a 'random shape' version of that described at the start of this article, in which *every* incoming item is accepted at a position uniformly distributed over feasible positions, until there are no feasible positions left. However, we do not consider such a model here.

The proof of Theorem 6.1 runs along mostly the same lines as those of Theorems 2.1 and 2.2, and we just give a sketch. As before, we may assume without loss of generality that $r_1 = 1$, so that there cannot be any overlap between items whose location is at an l_∞ distance more than 1 from one another. The appropriate oriented graph structure $\mathcal{G}$ on the points of $\mathcal{P}$ goes as follows. Draw an oriented edge from (X, T, Ξ) to (Y, U, Ξ') whenever $T < U$ and $\Phi(Y - X; \Xi, \Xi') = 0$. Using this graph structure we define $I(X, T, \Xi; A)$ in terms of the restriction of the graph to vertices in $A \times [0, \infty) \times \mathbb{F}$, and $I(X, T, \Xi)$ in terms of the entire graph, in a manner analogous to that used in Section 3. We then have

$$N^h(B; A) = \sum_{(X,T,\Xi)\in\mathcal{P}} h(\Xi) I(X, T, \Xi; A) \mathbf{1}_{\{X\in B\}},$$

and can also define

$$N_t^h(B) = \sum_{(X,T,\Xi)\in\mathcal{P}} h(\Xi) I(X, T, \Xi) \mathbf{1}_{\{X\in B, T\le t\}},$$

and

$$N_t(B) = \sum_{(X,T,\xi)\in\mathcal{P}} I(X, T, \xi) \mathbf{1}_{\{X\in B, T\le t\}}.$$

Also set $N^h(B) = \lim_{t\to\infty} N_t^h(B)$ and $N(B) = \lim_{t\to\infty} N_t(B)$.

Most of the modifications to the proof of Theorems 2.1 and 2.2 are straightforward. The definition of the blocking time J_z in the modified setting is

$$J_z := \inf\{t \ge 0 : N_t(Q_z^+) = N(Q_z^+)\}.$$

At time J_z, with probability 1 the cube Q_z is jammed, meaning in this case that the set V of pairs $(x, \zeta) \in Q_z \times \mathbb{F}$ such that $\Phi(x - X; \Xi, \zeta) = 1$ for all of the points (X, T, Ξ) of $\mathcal{P}$ such that $T \le J_z$ and $I(X, T, \Xi) = 1$, satisfies

$$(\text{Leb} \times P_1)(V) = 0.$$

This is the case because if it were not, then at some later time U after J_z there would be a Poisson arrival (X, U, Ξ') with $(X, \Xi') \in V$, and this item would either be accepted itself, contradicting the definition of J_z, or be blocked by some item arriving between times J_z and U, also contradicting the definition of J_z.

We can apply the Ergodic Theorem to get convergence of $N^h(A_n)/|A_n|$ to $\mathbb{E} N^h(Q_\mathbf{0})$, because if A is a union of cubes centred at integer lattice points, then $N^h(A)$ can be expressed as the sum over corresponding lattice points of a stationary functional of independent identically distributed Poisson processes on $Q_\mathbf{0} \times [0, \infty) \times \mathbb{F}$, indexed by lattice points. Likewise, $N_t^h(A_n)/|A_n|$ tends to $\mathbb{E} N_t^h(Q_\mathbf{0})$. We can approximate $N_t^h(A_n)$ by $N_t^h(A_n; A_n)$ and approximate $N_t^h(A_n; A_n)$ by $N^h(A_n; A_n)$ in much the same manner as before, and the remainder of the proof follows in the same way as in the earlier case.

7 Cooperative sequential adsorption

Cooperative sequential adsorption (CSA) is a generalization of RSA, in which an incoming item located at $x \in \mathbb{R}^d$ is accepted with a probability that depends on the local configuration of previously accepted items near to x. For example, in one-dimensional car-parking one can imagine that some of the more faint-hearted drivers might go away when confronted by a gap between cars that is only just big enough to contain another car. CSA models for deposition of molecules onto a surface are an attempt to incorporate intermolecular interactions more general than spatial exclusion. The Palásti RSA model is a special case of CSA in which the acceptance probability is a product over all accepted points of 'hard core' indicator functions.

In this section we indicate how to adapt the proof of Theorem 2.1 to obtain a similar LLN for more general CSA models, provided the acceptance probability function has a hard core component to it and is translation-invariant with finite range. In fact, many of the CSA models proposed in the scientific literature have infinite range effects; further work will be required to deal with these in a rigorous way. As in the preceding section, we could allow for the possibility of several different types of incoming item, but for the sake of clarity we restrict our attention to the following single-type CSA model.

Let $r_1 > 0$ be the range of interactions. Let $\Phi_0 \in (0,1]$, and for $k = 1, 2, 3, \dots$, let

$$\Phi_k : ([-r_1, r_1]^d)^k \to [0,1]$$

be a Borel measurable function that is permutation-invariant, meaning that for any permutation $\sigma : \{1, 2, \dots, k\} \to \{1, 2, \dots, k\}$, we have $\Phi_k(x_1, \dots, x_k) = \Phi_k(x_{\sigma(1)}, \dots, x_{\sigma(k)})$. The function Φ_k represents the acceptance probability as a function of relative displacement of existing items.

Given a bounded nonnull Borel target set $A \subset \mathbb{R}^d$, suppose items arrive sequentially at locations uniformly distributed over A. If an item arrives at location $x \in A$, and the already accepted items in $B_{r_1}(x)$ are at $x_1, \dots, x_k$, then the new item is accepted with probability $\Phi_k(x_1 - x, \dots, x_k - x)$. If there are no already accepted items in $B_{r_1}(x)$, then it is accepted with probability Φ_0.

Combining the functions Φ_k, let the function $\Phi(\mathcal{X})$, defined for all finite subsets $\mathcal{X}$ of $[-r_1, r_1]^d$, be given by

$$\Phi(\mathcal{X}) = \Phi_0 \quad \text{if} \quad \mathcal{X} = \emptyset;$$
$$\Phi(\mathcal{X}) = \Phi_k(x_1, \dots, x_k) \quad \text{if} \quad \mathcal{X} = \{x_1, \dots, x_k\}.$$

We assume Φ satisfies a 'hard core' condition, by which we mean that there exists $r_0 \in (0, r_1)$ such that

$$\Phi(\mathcal{X}) = 0 \quad \text{if} \quad \mathcal{X} \cap B_{r_0}(\mathbf{0}) \neq \emptyset.$$

This means that no incoming item is accepted if too close to an existing accepted item.

If the hard core condition is satisfied, then the ultimate number of accepted items in A is finite; let it be denoted $N(A; A)$ as usual. Following on from previous sections, the primary aim here is to give a law of large numbers for $N(A; A)$.

Theorem 7.1 *There is a constant $\lambda > 0$ such that if $(A_n)_{n \geq 1}$ is a sequence of bounded nonnull Borel subsets of $\mathbb{R}^d$ with $|\partial_r A_n|/|A_n| \to 0$ as $n \to \infty$ for all $r > 0$, then for all $p \in [1, \infty)$,*

$$\frac{N(A_n; A_n)}{|A_n|} \xrightarrow{L^p} \lambda. \tag{7.1}$$

It is also possible to obtain a point process convergence result along the lines of Theorem 2.2, but we do not go into details.

As for preceding results, it suffices to prove Theorem 7.1 in the case $r_1 = 1$, and from now on we assume this. Also as in previous cases, we wish to extend the CSA process to the whole of $\mathbb{R}^d$. This requires a modification of the previous graphical construction.

Assume that on a suitable probability space, we have a homogeneous Poisson process $\mathcal{P}$ of unit intensity on $\mathbb{R}^d \times [0,\infty)$, and each point (X,T) of the Poisson process $\mathcal{P}$ carries a mark $M(X,T)$ which is uniformly distributed on $[0,1]$, independently of the other arrivals. In other words, we are really taking a homogeneous Poisson process on $\mathbb{R}^d \times [0,\infty) \times [0,1]$, but we are viewing the third component M of each Poisson point (X,T,M) as a 'mark' assigned to the point (X,T). The first component X represents location and the second component T represents time of arrival, as usual.

Make the points of the Poisson process $\mathcal{P}$ on $\mathbb{R}^d \times [0,\infty)$ into the vertices of an oriented graph, denoted $\mathcal{G}$, by putting in an oriented edge $(X,T) \to (X',T')$ whenever $\|X' - X\| \leq 1$ and $T < T'$.

For $z \in \mathbb{Z}^d$, let the cubes Q_z and Q_z^+ be as defined in Section 4. For $x, y \in \mathbb{Z}^d$, let us say that y *is affected by* x *before time* t if there exists a (directed) path in $\mathcal{G}$ that starts at some Poisson point (X,T) and ends at some Poisson point (Y,U) with $X \in Q_x, Y \in Q_y$, and $U \leq t$. Let $E_t(x,y)$ denote the event that y is affected by x before time t.

Lemma 7.1 *There exists a constant $\delta_3 \in (0,1)$, such that for all $r > 0$ and all $z \in \mathbb{Z}^d$,*

$$P\left[\cup_{y \in \mathbb{Z}^d \setminus B_r(z)} E'_{\delta_3 r}(y,z)\right] \leq 2(3^{-r}). \tag{7.2}$$

Hence with probability 1, z is affected before time t by only finitely many y.

Proof. The proof of Lemma 3.1 carries through to the present case, to give us (7.2). The proof of the last assertion is the same as that of Corollary 3.1. ■

Suppose (X,T), is a point of $\mathcal{P}$, representing an incoming item at location X at time T with mark $M(X,T)$. Let us put $I(X,T;A) = 1$ if the item is accepted and $I(X,T;A) = 0$ if not. Let the rule for acceptance be as follows: set $I(X,T;A) = 1$ if $X \in A$ and

$$\begin{aligned} M(X,T) \leq \Phi(\{(X'-X) : (X',T') \in \mathcal{P}, \|X'-X\| \leq 1, \\ T' < T, I(X',T';A) = 1\}). \end{aligned} \tag{7.3}$$

Otherwise, set $I(X,T;A) = 0$. This clearly generates a realization of the CSA process with target set A as described earlier. We now give an equivalent generational construction of $I(X,T;A)$ which also applies when the target set is the whole of $\mathbb{R}^d$.

Given $A \subseteq \mathbb{R}^d$ (possibly the set $\mathbb{R}^d$ itself), let $\mathcal{P}_A = \mathcal{P} \cap (A \times [0,\infty))$. Let $\mathcal{G}_A$ be the restriction of $\mathcal{G}$ to vertices in $\mathcal{P}_A$. Let $G_0(A)$ be the set of roots of $\mathcal{G}_A$. After removing $G_0(A)$ from the vertex set of $\mathcal{G}_A$, let $G_1(A)$ be the set of roots of the remaining oriented graph. After removing $G_1(A)$ also, let $G_2(A)$ be the set of roots of the remaining graph. Repeating this process gives us a sequence $G_0(A), G_1(A), G_2(A), \ldots$ of disjoint subsets ('generations') of the vertex set of $\mathcal{G}_A$. By Lemma 7.1, they actually form a partition of $\mathcal{G}_A$.

Let (X,T) be a point of $\mathcal{P}_A$. Suppose first that $(X,T) \in G_0$, i.e. (X,T) is a root of $\mathcal{G}_A$; in this case, let $I(X,T;A)$ be equal to 1 if $M(X,T) \leq \Phi_0$, and be zero otherwise.

Now suppose, inductively, that (X,T) is in generation $G_m(A)$ and $I(X',T';A)$ has been determined for all (X',T') in generations $G_0(A), G_1(A), \ldots, G_{m-1}(A)$. Then put put $I(X,T;A) = 1$ if and only if

$$M \leq \Phi(\{(X'-X) : (X',T') \in \mathcal{P}_A, X' \in B_1(X), T' < T, I(X',T';A) = 1\}).$$

Since the sets $G_0(A), G_1(A), \ldots$ form a partition of $\mathcal{P}_A$, it can be seen that in the case where A is bounded, this definition of $I(X,T;A)$ agrees with that given at (7.3) earlier on.

In the case where $A = \mathbb{R}^d$, this definition of $I(X,T;A)$ remains valid, and we abbreviate $I(X,T;\mathbb{R}^d)$ to $I(X,T)$. Define

$$N_t(B) = \sum_{(X,T)\in\mathcal{P}} I(X,T)\mathbf{1}_{\{X\in B, T\le t\}}$$

and $N(B) = \lim_{t\to\infty} N_t(B)$.

For each $z \in \mathbb{Z}^d$, as in Section 4 define the 'jamming time'

$$J_z = \inf\{t \ge 0 : N_t(Q_z^+) = N(Q_z^+)\},$$

which is finite. At time J_z, with probability 1 the site z is jammed, meaning in this case that for Lebesgue-almost all $x \in Q_z$,

$$\Phi(\{X - x : (X,T) \in \mathcal{P}, X \in B_1(x), T \le J_z, I(X,T) = 1\}) = 0.$$

Indeed, if this were not the case then there would exist $\varepsilon > 0$ and a set $B \subset Q_z$ of strictly positive Lebesgue measure, such that for $x \in B$,

$$\Phi(\{X - x : (X,T) \in \mathcal{P}, X \in B_1(x), T \le J_z, I(X,T) = 1\}) > \varepsilon.$$

But then with probability 1, there would arrive a Poisson point (X', U) with $X' \in B$, $U > J_z$ and $M(X', U) < \varepsilon$, and the item this represented would be accepted unless there had been some added point in the cube Q_z^+ after time J_z, a contradiction.

With the above ingredients in place, the proof of Theorem 7.1 follows by very much the same argument as that of Theorem 2.1, and we do not give further details.

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