Lectures on Particle Systems and Epidemics

MATHEW D. PENROSE

Notes for lectures given at the Workshop on Percolation, Indian Statistical Institute, New Delhi, December 1996

1 First Passage Percolation

These lectures are concerned with three different models for spatial epidemics. Imagine organisms placed at the sites of the integer lattice $\mathbf{Z}^d$ ($d = 2$ is the most natural case to think of), which can be healthy or diseased. Infection can be passed between neighbouring sites.

In this lecture we look at **first passage percolation**. A strictly positive random variable X is specified, satisfying

$$E[e^{\theta X}] < \infty, \qquad \text{some } \theta > 0.$$

In first passage percolation, once a site $x \in \mathbf{Z}^d$ becomes infected, it never recovers. For each of its neighbours y it waits for a period of time X_{xy}, and then emits a germ in the direction of y; the X_{xy} are assumed to be independent copies of X. Such a germ causes y to become infected if it was not already infected, and otherwise has no effect.

We assume that at time zero, the origin becomes infected, with all other sites healthy. The epidemic will reach every site sooner or later, and the *rate* of spread is of interest. For $x \in \mathbf{Z}^d$ let $T(x)$ be the time it takes for x to first become infected, and let ξ_t be the region that is infected by time t, 'fattened out' by including the unit square around each infected site of $\mathbf{Z}^d$, to make a region in $\mathbf{R}^d$:

$$\xi_t = \{x + y : x \in \mathbf{Z}^d,\ y \in (-0.5, 0.5]^d,\ T(x) < t\}.$$

The main result of this lecture says that the infected region has an asymptotic deterministic 'shape'.

Theorem 1 (Shape Theorem) *There is a convex set $A \subset \mathbf{R}^d$ such that for $\varepsilon > 0$,*

$$\lim_{t \to \infty} P[(1 - \varepsilon)tA \subset \xi_t \subset (1 + \varepsilon)tA] = 1.$$

A stronger version of this theorem was proved by Cox and Durrett (1981).

For x and y in $\mathbf{Z}^d$, a *path* γ from x to y (written $\gamma : x \rightsquigarrow y$) is a sequence $x_0 = x, x_1, x_2, \ldots, x_n = y$ of distinct elements of $\mathbf{Z}^d$, with x_i a neighbour of x_{i-1} for each i (write $|\gamma|$ for n here). Let S_γ be be the time it would take for the infection to pass from x to y along the path γ, i.e.

$$S_\gamma = \sum_{i=1}^{n} X_{xy}$$

(remember, the X_{xy} are independent copies of X, defined for each ordered pair of neighbouring sites). For $x, y \in \mathbf{Z}^d$ set

$$T(x,y) = \inf_{\gamma : x \rightsquigarrow y} S_\gamma.$$

Then

$$T(x) = T(0,x).$$

With this construction, instead of a dynamically evolving epidemic we can think of the system as a static one based on the (X_{xy}), similar to percolation.

As a first step towards the shape theorem, we show $T(x)$ grows roughly linearly with $|x|$, with exponentially decaying error bounds. Here $|\cdot|$ is the l^1 norm on $\mathbf{R}^d$, i.e. for $x = (x^1, \ldots, x^d)$,

$$|x| = \sum_{i=1}^{d} |x^i|.$$

The lemma also shows that there are no 'explosions', i.e. $\{x : T(x) < t\}$ is finite for all t, almost surely.

Lemma 1 *There exist positive finite constants c, K, C, β, such that for all $x \in \mathbf{Z}^d$,*

$$P[T(x) < c|x|] \leq C \exp(-\beta|x|); \tag{1}$$

$$P[T(x) > K|x|] \leq C \exp(-\beta|x|). \tag{2}$$

Proof. The proof of (1) is based on counting paths, as in certain proofs of nontriviality of percolation thresholds. We have

$$P[T(x) < c|x|] = P\left[\cup_{\gamma:0\rightsquigarrow x}\{S_\gamma < c|x|\}\right] \leq \sum_{\gamma:0\rightsquigarrow x} P[S_\gamma < c|x|].$$

For $\theta > 0$, we have

$$P[S_\gamma < c|\gamma|] \leq \frac{E[e^{-\theta S_\gamma}]}{e^{-\theta c|\gamma|}} = \left(e^{\theta c} E[e^{-\theta X}]\right)^{|\gamma|}.$$

Let $\alpha < 1/(2d)$. Take $\theta > 0$ with $E[e^{-\theta X}] < \alpha/2$ and c with $e^{\theta c} < 2$. Then $P[S_\gamma < c|\gamma|] \leq \alpha^{|\gamma|}$, and hence

$$\begin{aligned} P[T(x) < c|x|] &\leq \sum_{n=|x|}^{\infty} \sum_{\gamma : |\gamma| = n} P[S_\gamma < c|\gamma|] \\ &\leq \sum_{n=|x|}^{\infty} (2d)^n \alpha^n = \frac{(2d\alpha)^{|x|}}{1 - 2d\alpha}. \end{aligned}$$

This gives us (1). To prove (2), take $\gamma : 0 \rightsquigarrow x$ with $|\gamma| = |x|$. For $\theta > 0$,

$$\begin{aligned} P[T(x) > K|x|] &\leq P[S_\gamma > K|x|] \\ &\leq \frac{Ee^{\theta S_\gamma}}{e^{\theta K|x|}} = \left(\frac{Ee^{\theta X}}{e^{\theta K}}\right)^{|x|}. \end{aligned}$$

By taking $K > E[X]$ and θ suitably small we get (2). □

Next we show the infection spreads at a deterministic asymptotic rate in any fixed direction.

Lemma 2 *Let $x \in \mathbf{Z}^d$. Then there is a number $\mu = \mu(x)$ such that*

$$\frac{T(nx)}{n} \to \mu \quad \text{in } L^1. \tag{3}$$

Proof. The proof is based on the *subadditive* property:

$$T(x, y) \leq T(x, z) + T(z, y), \quad \forall x, y, z \in \mathbf{Z}^d,$$

and the *stationary* property: for any $y \in \mathbf{Z}^d$,

$$(T(0, x), x \in \mathbf{Z}^d) \text{ has the same distribution as } (T(y, y + x), x \in \mathbf{Z}^d).$$

The result (3) is a special case of the *Subadditive Ergodic Theorem.* For a full statement of this, see e.g. Liggett (1985). The convergence in (3) also holds a.s. but we shall not prove this.

Set $a_n = E[T(nx)]$. The limit μ is given by

$$\mu = \inf_n (n^{-1} a_n).$$

We first prove that $\lim_{n\to\infty}(a_n/n) = \mu$. It suffices to show that for any fixed m we have $\limsup(a_n/n) \leq a_m/m$.

Set $S_{m,n} = T(mx, nx)$. For $n \in \mathbf{Z}_+$ write $n = km + l$ with $k, l \in \mathbf{Z}$ and $0 \le l < m$. By subadditivity

$$\frac{T(nx)}{n} = \frac{S_{0,n}}{n} \le \frac{S_{0,m} + S_{m,2m} + \cdots + S_{(k-1)m,km}}{km + l} + \frac{S_{km,n}}{n}. \tag{4}$$

Taking expectations and using stationarity,

$$\frac{a_n}{n} \le \frac{ka_m}{km} + (a_l/n)$$

and the second term tends to zero since a_l runs through only finitely many values, so that $\limsup(a_n/n) \le (a_m/m)$, which shows $(a_n/n) \to \mu$ as claimed.

Going on to the proof of L^1 convergence (3), we use (4) again. First, we claim that by stationarity

$$\lim_{k\to\infty} \frac{S_{0,m} + S_{m,2m} + \cdots + S_{(k-1)m,km}}{k} = E[S_{0,m}] \quad \text{a.s.} \tag{5}$$

Indeed, by Birkhoff's individual ergodic theorem, the limit exists and has mean $E[S_{0,m}]$, We claim the limit is in the tail σ-algebra of the X_{yz}'s, and so by the Kolmogorov $0-1$ law is constant, giving (5). To see the limit is in the tail σ-algebra, note that changing a single X_{yz} causes only finitely many of the $S_{jm,(j+1)m}$'s to change, a.s. This is because by Lemma 1 and Borel-Cantelli, there exists $c > 0$ such that for large enough j we have

$$T(jmx, (j+1)mx) < cj, \quad \text{and} \quad T(jmx, y) > cj,$$

which implies that changing $X_{y,z}$ does not affect $T(jmx, (j+1)mx)$ (alternatively, it can be shown that the limit in (5) is translation-invariant and therefore constant - see Lemma 3.2 of Meester (1996)).

The second term on the right in (4) tends to zero a.s. by Borel-Cantelli, since for $\varepsilon > 0$ and each fixed $l \in \{0, 1, 2, \ldots, m-1\}$ we have

$$\sum_n P[n^{-1}S_{0,l} \ge \varepsilon] < \infty,$$

since $S_{0,l}$ has finite mean.

Thus by (4), $\limsup T(nx)/n \le a_m/m$ almost surely, and since m is arbitrary we have $\limsup T(nx)/n \le \mu$, i.e.

$$\lim_{n\to\infty} \left(\frac{T(nx)}{n} - \mu\right)^+ = 0 \quad \text{a.s.} \tag{6}$$

For each n take $\gamma(n) : 0 \rightsquigarrow nx$ with $|\gamma(n)| = n|x|$. Then $ES_{\gamma(n)}$ and $Var(S_{\gamma(n)})$ grow linearly with n, so there exists K such that

$$E[T(nx)^2] \le E[S^2_{\gamma(n)}] \le Kn^2.$$

Hence (since $0 \leq (\frac{T(nx)}{n} - \mu)^+ \leq \frac{T(nx)}{n}$ trivially), we have

$$E\left(\left(\left(\frac{T(nx)}{n} - \mu\right)^+\right)^2\right) \leq K,$$

so a routine use of Cauchy-Schwarz and (6) gives

$$\lim_{n\to\infty} E\left[\left(\frac{T(nx)}{n} - \mu\right)^+\right] = 0.$$

We proved earlier that $\lim_{n\to\infty} E[T(nx)/n] = \mu$, and therefore the mean of the negative part also goes to zero, and (3) follows. □

We extend the definition of $T(x)$ to the whole of $\mathbf{R}^d$ by

$$T(x) = \inf\{t \geq 0 : x \in \xi_t\}.$$

Lemma 3 *For all $x \in \mathbf{R}^d$, there exists $\mu(x) \geq 0$ such that*

$$\lim_{t\to\infty} E\left[\left|\frac{T(tx)}{t} - \mu(x)\right|\right] = 0. \tag{7}$$

Moreover, $\mu(x) > 0$ for $x \neq 0$.

Of course for $x \in \mathbf{Z}^d$ the limit $\mu(x)$ must be the limit μ established in (3) for t running through the integers.

Proof of Lemma 3. For any u and v in $\mathbf{Z}^d$, by subadditivity

$$|T(u) - T(v)| \leq T(u, v) + T(v, u)$$

and this upper bound is bounded by the sum of $2|u - v|$ independent copies of X, so that for x and y in $\mathbf{R}^d$, and $K > E[X]$,

$$\|T(x) - T(y)\|_1 \leq 2K|x - y| + dK. \tag{8}$$

First take x with rational coordinates, take $m \in \mathbf{Z}$ with $mx \in \mathbf{Z}^d$, so that $\mu(mx)$ is given in (3). Then setting $t = km + l$ with k an integer and $0 \leq l < m$, we have

$$\frac{T(tx)}{t} = \frac{T(kmx)}{t} + \frac{T(tx) - T(kmx)}{t}.$$

The first term on the right converges in L^1 to $\mu(mx)/m$ by (3), while the second term converges to zero by (8), giving (7) for x with rational coordinates.

The proof of (7) for general x is left as an exercise (use (8) again). The assertion that $\mu > 0$ for $x \neq 0$ follows from (1). □.

Proof of the Shape Theorem. It is easy to see from subadditivity and (8) that $\mu(x+y) \leq \mu(x) + \mu(y)$, and $\mu(cx) = |c|\mu(x)$ for $c \in \mathbf{R}$. Thus μ is a norm on $\mathbf{R}^d$. Let A be the unit ball of that norm, which is necessarily convex.

For $r > 0$, let $D(x,r) = \{y : |y-x| \leq r\}$, the r-ball centred at x (in the l^1 norm on $\mathbf{R}^d$: the D stands for 'diamond').

We first show that

$$P[(1-\varepsilon)tA \subset \xi_t] \to 1. \tag{9}$$

Let $\delta = \varepsilon/(2K)$, with K as in Lemma 1. By compactness we can (and do) take points $x_1, \ldots, x_k \in (1-\varepsilon)A$ such that $(1-\varepsilon)A \subset \cup_{i=1}^k D(x_i, \delta/2)$. For each i we have $\mu(x_i) \leq 1-\varepsilon$ (definition of A), so

$$P\left[\frac{T(tx_i)}{t} < 1 - \varepsilon/2\right] \to 1. \tag{10}$$

Also,

$$P[\cup_{y \in D(tx_i, t\delta/2)}\{T(tx_i, y) > \varepsilon t/2\}] \leq \sum_{z \in \mathbf{Z}^d \cap D(0,t\delta)} P[T(0,z) > K\delta t],$$

which tends to zero as $t \to \infty$, by Lemma 1 and the Dominated Convergence Theorem. Thus

$$P[\cap_{y \in D(tx_i, t\delta/2)}\{T(tx_i, y) \leq \varepsilon t/2\}] \to 1,$$

and by combining this with (10) and using subadditivity, we get

$$P[D(tx_i, t\delta/2) \subset \xi_t] \to 1,$$

hence (9).

For the other inclusion, let F_t be the event $\{\xi_t \subset (1+\varepsilon)tA\}^c$. We wish to show $P[F_t] \to 0$. Assume $\varepsilon < 1$. Let δ be as above.

By compactness, we can pick points, again denoted $x_1, \ldots, x_k$, in $2A \setminus (1+\varepsilon)A$, such that $2A \setminus (1+\varepsilon)A \subset \cup_{i=1}^k D(x_i, \delta/2)$.

By the connectivity of ξ_t, the occurrence of F_t implies $\xi_t \cap (2tA \setminus (1+\varepsilon)tA)$ is nonempty, which implies there is some $z \in \mathbf{Z}^d \cap D(tx_i, t\delta)$ for some $i \leq k$, with $T(z) \leq t$. But by Lemma 1,

$$\sum_{\{z \in \mathbf{Z}^d : |z - tx_i| \leq \delta t\}} P[T(z, tx_i) > t\varepsilon/2] \to 0,$$

and hence

$$P[\cup_{i=1}^k \{T(tx_i) < t + t\varepsilon/2\} | F_t] \to 1. \tag{11}$$

However, for each i, since $\mu(x_i) \geq 1 + \varepsilon$,

$$P\left[\frac{T(tx_i)}{t} \leq 1 + \varepsilon/2\right] \to 0.$$

Combining this with (11), we get a contradiction unless $P[F_t] \to 0$, as required. $\square$

2 The Contact Process

The contact process is like first passage percolation, but with particles recovering at rate 1, i.e. an infected particle remains so for an exponential time with mean 1, and then recovers. Also, the X_{xy} are taken to be exponential with mean $1/\lambda$, so the evolving set of infected sites is a continuous time Markov chain $\xi_t, t \geq 0$, with state space, denoted $\mathcal{S}$, the space of all subsets of $\mathbf{Z}^d$. The dynamics are: for $A \subset \mathbf{Z}^d$ and $x \in \mathbf{Z}^d$,

$$P[\xi_{t+h} = A \cup \{x\} | \xi_t = A] = h\lambda |\mathcal{N}(x) \cap A| + o(h) \quad \text{if } x \notin A;$$

$$P[\xi_{t+h} = A \setminus \{x\} | \xi_t = A] = h + o(h) \quad \text{if } x \in A.$$

Here $|\cdot|$ denotes cardinality, and $\mathcal{N}(x)$ denotes the set of neighbours of x in $\mathbf{Z}^d$. Thus each infected site infects its neighbours at rate λ and recovers at rate 1.

Think of ξ_t as a set of particles (representing infected sites) which 'give birth' at unoccupied neighbours at rate λ and die at rate 1. The process $|\xi_t|$ would be a continuous time branching process, except for the fact that births at already occupied sites are suppressed (the 'excluded volume effect').

The process can be represented graphically. On a suitable probability space, suppose there exist independent Poisson processes $\mathcal{P}_{xy}$ of rate λ for each pair x, y of neighbouring sites in $\mathbf{Z}^d$, and Poisson processes $\mathcal{Q}_x$ of rate 1 for each $x \in \mathbf{Z}^d$. The arrival times of $\mathcal{P}_{xy}$ represent the instants at which 'germs' are emitted from x in the direction of y (if x is not infected at such an instant, think of x as emitting a 'virtual germ' that does not affect y). The arrival times of $\mathcal{Q}_x$ are the instants at which x recovers (if infected just before such an instant).

We make a graph in $d+1$ dimensions (the extra dimension represents 'time'), as follows. For each $x \in \mathbf{Z}^d$, draw the 'vertical' line $\{x\} \times [0, \infty)$, oriented in the direction of increasing 'time', and for each neighbouring pair $\{x, y\}$ and each arrival time Γ of the Poisson process $\mathcal{P}_{xy}$, draw an oriented line from (x, Γ) to (y, Γ). For each arrivial time Γ of $\mathcal{Q}_x$, remove the points (x, Γ) from the graph.

For $x \in \mathbf{Z}^d$, let ξ_t^x be the set of points $y \in \mathbf{Z}^d$ such that (y, t) can be reached from $(x, 0)$ along the oriented edges of the resulting graph described above (i.e. along the vertical and added edges but not passing through the removed points). Let $\xi_t^A = \cup_{x \in A} \xi_t^x$. It can be seen that $(\xi_t^A, t \geq 0)$ is a realization of the contact process with initial state A (we allow A infinite). Note that in this construction gives coupled versions of the processses ξ_t^A for all initial states $A \subset \mathbf{Z}^d$ simultaneously. If we wish to emphasise that the 'birth rate' parameter is λ we'll write $\xi_t^{A,\lambda}$.

Two monontonicities are clear, namely

$$(A \subset B) \Rightarrow \xi_t^A \subset \xi_t^B \tag{1}$$

and

$$(\lambda < \mu) \Rightarrow \exists \text{ a coupling with } \xi_t^{A,\lambda} \subset \xi_t^{A,\mu}, \quad \forall t \geq 0. \tag{2}$$

(1) is the so-called 'attractive' property of the contact process: the graphical representation gives a coupling in which the contact process starting on a bigger set continues to dominate the contact proces starting on a smaller set, for all time.

The empty set $\emptyset$ is an absorbing state for the contact process (since particles can only be born at the neighbours of existing particles), and we say the contact process *dies out* (or just *dies*) if $\xi_t = \emptyset$ for some $t \geq 0$; otherwise we say the process *survives*. Note that for any two nonempty finite sets A, B,

$$P[\xi_t^A \text{ survives }] > 0 \Leftrightarrow P[\xi_t^B \text{ survives }] > 0,$$

since there is a non-zero probability that $\xi_1^A = B$, and vice versa.

By the monotonicity (2), $P[\xi_t^{A,\lambda} \text{ survives }]$ is nondecreasing in λ, and so there exists $\lambda_c \in [0, \infty]$ such that

$$(\lambda < \lambda_c) \Rightarrow P[\xi_t^A \text{ survives}] = 0 \qquad \text{for all finite } A \subset \mathbf{Z}^d,$$
$$(\lambda > \lambda_c) \Rightarrow P[\xi_t^A \text{ survives}] > 0 \qquad \text{for all nonempty finite } A \subset \mathbf{Z}^d.$$

It is now known (but not proved here) that at $\lambda = \lambda_c$ the process ξ_t^A dies a.s. for each finite A. See Bezuidenhout and Grimmett (1990).

Proposition 1 $0 < \lambda_c < \infty$.

Proof. As mentioned before, $|\xi_t|$ can be viewed as a branching process with some of the births suppressed. There is a coupling with $|\xi_t| \leq \zeta_t$ for all t, with ζ_t a branching process with birth rate $2d\lambda$ per particle and death rate 1 per particle. If $\lambda < 1/(2d)$, the branching process (ζ_t) dies out a.s., hence so does (ξ_t). Thus $\lambda_c \geq 1/(2d) > 0$.

To prove $\lambda_c < \infty$, it suffices to consider $d = 1$. We compare with *oriented site percolation* on the lattice

$$\mathcal{L} = \{(n, m) \in \mathbf{Z} \times \mathbf{Z} : m \geq 0, n + m \text{ even}\}$$

with oriented edges from each (n, m) to $(n \pm 1, m + 1)$. There exists $p_0 < 1$, such that $p_0 < p < 1$, if each site of $\mathcal{L}$ is independently open with probability p, then there is an infinite open oriented path from 0 with strictly positive probability. The proof of this is similar to that of similar results for ordinary percolation.

Pick $\eta > 0$. We deem site (n, m) of $\mathcal{L}$ to be open if

(i) there are no 'deaths' at site n (i.e. arrivals of $\mathcal{Q}_n$) between times $(m - 1)\eta$ and $(m + 1)\eta$:

(ii) There are 'attempted births' from n to $n + 1$ and also to $n - 1$ (i.e. arrivals of $\mathcal{P}_{n,n+1}$ and of $\mathcal{P}_{n,n-1}$) between times $m\eta$ and $(m + 1)\eta$.

Then it can be seen that (i) the status of the different sites of $\mathcal{L}$ are independent; (ii) the existence of an infinite oriented open path in $\mathcal{L}$ implies the survivial of ξ_t^0;

and (iii) by choosing η suitably small and λ suitably big, the probability that each site is open can be made close to 1. $\square$

Invariant measures. Let μ be a probability measure on the space $\mathcal{S}$ of subsets of $\mathbf{Z}^d$. It is *invariant* if

$$T_t\mu = \mu \quad \forall\, t \geq 0$$

where T_t is the semigroup of the contact process, i.e.

$$T_t\mu(\cdot) = \int_{\mathcal{S}} P[\xi_t^A \in \cdot]d\mu(A),$$

meaning that if μ is the distribution of ξ_0 then $T_t\mu$ is the distribution of ξ_t.

The invariant measures are of interest for the contact process (indeed for any Markov process) as the large-t limiting distribution of the process. A process is *ergodic* if it has just a single invariant measure. One invariant measure is the trivial measure $\mu = \delta_\emptyset$, the point mass on the absorbing state $\emptyset$. The contact process is ergodic if and only if $\delta_\emptyset$ is the only invariant measure.

We consider now the 'maximal' contact process $\xi_t^{\mathbf{Z}^d}$ with initial state $\mathbf{Z}^d$, i.e. with *all* sites initially occupied/infected. By the attractive property, we have $\xi_t^{\mathbf{Z}^d} \supset \xi_t^A$ for any possible initial state A. Clearly the distribution of $\xi_t^{\mathbf{Z}^d}$ is translation-invariant, and for all $x \in \mathbf{Z}^d$ we have

$$P[x \in \xi_t^{\mathbf{Z}^d}] = P[0 \in \xi_t^{\mathbf{Z}^d}] =: \rho_t.$$

Thus ρ_t is the density of particles for $\xi_t^{\mathbf{Z}^d}$. We have

$$\rho_t \text{ is nonincreasing in } t. \tag{3}$$

This can be proved directly using the Markov and attractive properties (exercise) but we'll see an indirect proof later. By (3), $\lim_{t\to\infty} \rho_t$ exists. When this limit is *not* zero, the limiting distribution $\nu := \lim_{t\to\infty} T_t\delta_{\mathbf{Z}^d}$ is a second invariant measure besides $\delta_\emptyset$. The measure ν is called the *upper invariant measure*. When the limit *is* zero, we have $P[x \in \xi_t^A] \to 0$ for all A, and $\delta_\emptyset$ is the *only* invariant measure, i.e. the process is ergodic (since if μ is invariant then $\mu = T_t\mu$ which converges weakly to $\delta_\emptyset$ as $t \to \infty$, so that $\mu = \delta_\emptyset$).

We now show that ergodicity can be related to survival of the contact process starting from a single point, by the following result.

Proposition 2

$$\lim_{t\to\infty} \rho_t = P[\xi_t^0 \text{ survives}]. \tag{4}$$

Hence the contact process is ergodic, if and only if $\lambda \leq \lambda_c$.

Proof. The contact process has the following property known as *self-duality*: for any $A, B \subset \mathbf{Z}^d$,

$$P[\xi_t^A \cap B \neq \emptyset] = P[\xi_t^B \cap A \neq \emptyset]. \tag{5}$$

To see this, look at the graphical representation. The *reversed graphical representation* has the direction of all arrows reversed, and can be seen to be a graphical representation of a version of the contact process running 'backwards in time'.

The left side of (5) is the probability that there is a path from $A \times \{0\}$ to $B \times \{t\}$ in the original graph, while the right side of (5) is the probability that there is a path from $B \times \{t\}$ to $A \times \{0\}$ in the reversed graph (both paths avoiding removed points), i.e. the same event!

Setting $A = \mathbf{Z}^d$ and $B = \{0\}$ in (5), we see that ρ_t equals the probability that ξ_t^0 survives for time t. This gives us (3), and by taking $t \to \infty$ we also get (4). $\square$

We conclude with two striking results about the contact process, whose proofs are beyond the scope of these lectures. The first is analogous to the shape theorem given earlier for first passage percolation.

Theorem 1 (Shape Theorem) *Let $\lambda > \lambda_c$. Set $T(x) = \inf\{t \geq 0 : x \in \xi_t^0\}$ and*

$$H_t = \{x + y : x \in \mathbf{Z}^d, y \in (-0.5, 0.5]^d, T(x) \leq t\},$$

the 'fattened out' set of sites infected before time t, then there exists convex $A \subset \mathbf{R}^d$ such that for $\varepsilon > 0$,

$$P[(1-\varepsilon)tA \subset H_t \subset (1+\varepsilon)tA \text{ eventually}] = P[\xi_t^0 \text{ survives}]$$

Finally we note that when $\lambda > \lambda_c$, *all* invariant measures are convex combinations of the upper invariant measure ν and the trivial measure $\delta_\emptyset$. In fact we have the following.

Theorem 2 (Complete Convergence Theorem) *For $A \subset \mathbf{Z}^d$, let π_A denote the probability that ξ_t^A survives. Then in the limit $t \to \infty$, the distribution $T_t\delta_A$ of ξ_t^A converges weakly to $\pi_A\nu + (1-\pi_A)\delta_\emptyset$.*

Both the above theorems are given in Bezuidenhout and Grimmett (1990).

Remark For an explanation of the concept of weak convergence of measures on $\mathcal{S}$, used at various points in the above, see Liggett (1985). Note that $\mathcal{S}$ can be identified with the space of functions from $\mathbf{Z}^d$ to $\{0, 1\}$ by identifying each set with its indicator function.

3 The epidemic with removal

In this model for an epidemic or forest fire, sites of $\mathbf{Z}^d$ can be in states 0 (susceptible), 1 (infected) or 2 (immune). An infected site emits germs towards its neighbours (as for the contact process), but at the end of its infected period it becomes permanently immune (rather than returning to susceptiblity as in the contact process). Thus:

- An infected site emits germs at the times of a Poisson process of rate $2d\lambda$.
- A germ emitted from site x alights on one of its $2d$ neighbours selected uniformly at random, and causes this neighbour to become infected if it was previously susceptible. Otherwise it has no effect.
- Once it becomes infected, a site remains infected for a random period of time Y_x with distribution function F, and then becomes immune. We assume $F(0) = 0$.
- Initially, the origin is infected and other sites are susceptible.

The state ξ_t of the process at time t is a function $\xi_t : \mathbf{Z}^d \to \{0, 1, 2\}$, telling us which sites are susceptible, infected, immune. The process (ξ_t) is not Markov unless F is an exponential distribution. Note that if the Y_x are taken to be $+\infty$ almost surely, the process reduces to first passage percolation.

We define the infection time $T(x)$ at site x by

$$T(x) = \inf\{t : \xi_t(x) = 1\},$$

with the infimum taken to be $+\infty$ if x is never infected. We say the epidemic *dies out* if $\{x : T(x) < \infty\}$ is finite, and the epidemic *survives* if $\{x : T(x) < \infty\}$ is infinite. Note that some sites never become infected because of the formation of immune 'barriers'.

Of those germs emitted from x towards y, only the *first* one emitted after time $T(x)$ can possibly infect y (if it does not, then y is not susceptible at that time so will not be susceptible at later times either). Hence, the process ξ_t can be constructed based on mutually independent random variables $Y_x (x \in \mathbf{Z}^d)$ with distribution function F and G_{xy} $(x \in \mathbf{Z}^d, y \in \mathcal{N}(x))$ with the exponential distribution with mean $1/\lambda$. The G_{xy} represents the amount of time from $T(x)$ until the instant x first emits a germ towards y.

We can make a 'static' graphical representation of the 'dynamic' epidemic, as follows. For neighbouring x and y in $\mathbf{Z}^d$ we say the directed edge (x, y) of the integer lattice is *open* if $G_{x,y} < Y_x$. For any two sites u, v we write $u \to v$ if there is a path of open edges from u to v. Let C_0 denote the 'cluster at the origin' $\{x : 0 \to x\}$.

Lemma 1 $C_0 = \{x \in \mathbf{Z}^d : T(x) < \infty\}$

Proof. If $T(x) < \infty$, then there is a path of infections from 0 to x, and for each edge (y, z) in the path we have $G_{y,z} < Y_y$, so (y, z) is open, and thus $x \in C_0$.

For the inclusion the other way, suppose that $T(y) < \infty$ and (y, z) is open, i.e. $G_{y,z} < Y_y$. Then y tries to infect z at time $T(y) + G_{y,z}$. It succeeds unless z is already in state 1 or 2 by then, and either way $T(z) < \infty$.

So, if $x \in C_0$, there is an open path from 0 to x, with each site in the path becoming infected at some time; thus $T(x) < \infty$. $\square$

We can couple two epidemic models with the same F but with different Poisson parameters $\lambda < \mu$, as follows. We can couple exponential random variables $G^{\lambda}_{x,y}$ and $G^{\mu}_{x,y}$ in such a way that $G^{\mu}_{x,y} < G^{\lambda}_{x,y}$ a.s. If we make this coupling for each pair of neighbours x, y, and use the same set of random variables Y_x in each model, then every open edge in the λ model is open in the μ model. It follows that with this coupling, $C^{\lambda}_0 \subset C^{\mu}_0$. Thus using Lemma 1 we have the monontonicity

$$P[\xi^{\lambda}_t \text{ survives}] \text{ is nondecreasing in } \lambda.$$

Thus, there is a critical value $\lambda_c \in [0, \infty]$ such that for $\lambda < \lambda_c$ the epidemic dies out a.s., while for $\lambda > \lambda_c$ the epidemic survives with positive probability. The nontriviality of λ_c is given by

$$0 < \lambda_c < \infty, \quad \text{if } d \geq 2.$$

To see this, let us deem sites $x \in \mathbf{Z}^d$ to be *occupied* if $Y_x > \max_{y \in \mathcal{N}(x)} G_{x,y}$, and *vacant* if $Y_x < \min_{y \in \mathcal{N}(x)} G_{x,y}$, so that we allow the possibility that a site is neither occupied nor vacant.

For any path of occupied sites starting at the origin, we have $x \in C_0$ for all sites x in the path. Also, if λ is sufficiently large, then the probaility that a givien site is occupied exceeds the critical value for Bernoulli site percolation on $\mathbf{R}^d$, so there is an infinite occupied path with positive probability. Therefore $\lambda_c < \infty$.

If site x is vacant then there is no open edge of the form (x, y). As $\lambda \to 0$ the probability that a site is closed tends to 1. If this probability is sufficiently large the probability that non-vacant sites percolate is zero. Thus $\lambda_c > 0$.

For $d = 2$ we have yet another shape theorem for the supercritical epidemic.

Theorem 1 (Shape Theorem) *Let $d = 2$. Let $V_t = \{x : \xi_t(x) = 2\}$, the set of immune sites at time t. Let D_t be the set of infected sites at time t. Then there exists a convex set A such that for $\varepsilon > 0$,*

$$P[C_0 \cap t(1-\varepsilon)A \subset V_t \subset C_0 \cap t(1+\varepsilon)A] \to 1$$

and

$$P[D_t \subset t(1+\varepsilon)A \setminus t(1-\varepsilon)A] \to 1.$$

This result was proved by Cox and Durrett (1988). It is related to the main result on first passage percolation, given in Lecture 1 above. Indeed, if we set $X_{xy} = Y_{x,y}$ when edge (x, y) is open, and $X_{xy} = \infty$ otherwise, then if we follow the notation of that lecture by defining S_γ to be the sum of the X_{xy}'s along a path γ, and

$$T(x, y) = \inf_{\gamma : x \rightsquigarrow y} S_\gamma$$

then we have $T(x) = T(0, x)$ as in that section.

In the present setting there is some local dependence between the variables X_{xy}, but the main source of difficulty in trying to use the proof given earlier is the fact that $P[T(x, y) = \infty] > 0$ so $E[T(x, y)] = \infty$. The trick to get around this is to use the open loops around x and y, which exist in the supercritical regime by a form of the RSW lemma, to define a modification of $T(x, y)$ which is subadditive and has finite mean. This trick is essentially 2-dimensional.

Bounds on λ_c. We now obtain some numerical bounds on λ_c by expressing the epidemic as a *locally dependent directed percolation model* (LDDPM). For a general formulation of a LDDPM, let $\mathcal{G}$ be a dircted graph, with vertex-set V and edge-set E (here, all edges are *directed*). Let $\Omega := \{0, 1\}^E$ and for $\omega \in \Omega$, let us say an edge $e \in E$ is *open* if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. For any vertex $x \in V$ let $\mathcal{E}_x$ be the σ-algebra generated by the status of edges emanating from x, i.e. $\mathcal{E}_x$ is the σ-algebra generated by all random variables of the form $\omega(e)$, e an edge in E emanating from x.

A LDDPM on this graph is a probability measure P on Ω such that the the σ-algebras $\mathcal{E}_x$, $x \in V$ are mutually independent under P. Clearly the graphical representation of the epidemic above is a LDDPM on $\mathbf{Z}^d$.

For $x \in V$, let $\mathcal{N}(x)$ be the set of neighbours, i.e. the set of $y \in V$ such that $(x, y) \in E$. Given a LDDPM on $\mathcal{G}$, for $x \in V$ and $A \subset \mathcal{N}(x)$ set

$$\chi(x, A) = P[\text{ all edges of } A \text{ are closed }]$$

Lemma 2 *Suppose P_1 and P_2 are LDDPMs on $\mathcal{G}$ such that (with the obvious notation), for all $x \in V$ and $A \subset \mathcal{N}(x)$ we have*

$$\chi_1(x, A) \geq \chi_2(x, A).$$

Suppose that $\mathcal{B}$ is a finite collection of finite self-avoiding paths, and let B denote the event that some path in $\mathcal{B}$ is open. Then $P_1(B) \leq P_2(B)$.

Proof. Without loss of generality assume $\mathcal{G}$ is a finite graph. Also, asusume for now that the joint distribution of the edges other than those emanating from x is the same under P_1 and P_2. Let C be a configuration on all edges other than those emanating from x. If C includes a path in $\mathcal{B}$, then $P_1[B|C] = P_2[B|C] = 1$. If not, let $A(C)$ be the set (possibly empty) of edges from x whose addition would complete a path in $\mathcal{B}$. Note that no path in $\mathcal{B}$ has more than one such edge (since the paths are self-avoiding). Then, by the local dependance structure,

$$\begin{aligned} P_1[B|C] &= 1 - \chi_1(x, A(C)) \\ &\leq 1 - \chi_2(x, A(C)) = P_2[B|C]. \end{aligned}$$

Summing over all configurations C on edges other than those from x and using our initial extra assumption, we get

$$\begin{aligned} P_1[B] = \sum_C P_1[C] P_1[B|C] &= \sum_C P_2[C] P_1[B|C] \\ &\leq \sum_C P_2[C] P_2[B|C] = P_2[B]. \end{aligned}$$

Now drop the extra assumption made above. Start with the probablity measure P_1, and modify site by site until you get to P_2. At each step, the number $P[B]$ does not decrease. $\square$

We now return to the epidemic model on $\mathbf{Z}^d$ (viewed as a *directed* graph). Let p be the open bond density, i.e.

$$p = \int_0^\infty (1 - e^{-\lambda t}) dF(t).$$

Let P_{bond} be the LDDPM on $\mathbf{Z}^d$ in which each bond is independently open with probability p. Let P_{site} be the LDDPM on $\mathbf{Z}^d$ in which the bonds from x are all open with probability p and all closed with probability $1-p$ (with bonds from distinct sites independent, of course). Let simply P refer to the LDDPM associated with the original epidemic model. Thus we have three distinct LDDPMs on $\mathbf{Z}^d$, each with open bond density p. The comparison lemma above gives us the following.

Lemma 3 $P_{site}[|C_0| = \infty] \leq P[|C_0| = \infty] \leq P_{bond}[|C_0| = \infty]$.

Proof. For the LDDPM P, the bonds from x are positively correlated. This fact is a version of the FKG inequality, and gives us the second inequality below. For $A \subset \mathcal{N}(x)$,

$$\begin{aligned} \chi_{site}(x; A) &= (1-p) \\ &\geq \chi(A) \\ &\geq (1-p)^{|A|} = \chi_{bond}(x, A). \end{aligned}$$

Now apply the previous lemma, using the finite approximation

$$P[|C_0| = \infty] = \lim_{n\to\infty} P[\exists \text{ open path } \gamma : 0 \rightsquigarrow \{x : |x| = n\}]. \square$$

The LDDPM P_{site} is simply site percolation on $\mathbf{Z}^d$. However, P_{bond} is slightly different from the usual bond percolation model, here denoted P'_{bond}. For $y \in \mathcal{N}(x)$ there is a single undirected edge for P'_{bond}, whereas in P_{bond} there are two directed edges (x, y) and (y, x), with independent open/closed status. Nevertheless we have the folowing:

Lemma 4 $P_{bond}[|C_0| = \infty] = P'_{bond}[|C_0| = \infty]$.

Proof. Let G be a finite undirected graph. Let S and T be subsets of the vertex set of G. Let $S \to T$ denote the event that there is an open path from some site in S to some site in T. We claim that

$$P'[S \to T] = P[S \to T], \tag{1}$$

where P' refers to ordinary bond percolation on G with parameter p, while P refers to the independent LDDPM with parameter p on the graph obtained by replacing each edge of G by two directed edges (one each way). By a finite approximation argument, (1) suffices to give the result.

To prove (1) we start with the bond percolation model P', and change the model one edge at a time. At each step we replace a single undirected edge by two directed edges, until we end up with P. We claim that at each step the probability that there is an open path from S to T does not change.

To see this last claim, we have to compare two graphs, which differ at just a single edge $\{x, y\}$ (which is a single undirected edge in one case and is two directed edges in the other). We condition on the configuration on all other edges. Given a configuration C on all these other edges, $P[S \to T|C]$ is 0 in both models or 1 in both models, *except* when one of the following occurs (in the following the notation $S \to_C x$, etc. refers to the existence of open paths in the configuration C, i.e. not using the edge between x and y):

- $S \to_C x$ and $y \to_C T$, but not $S \to_C T$ (and hence, not $S \to_C y$):
- $S \to_C y$ and $x \to_C T$ but not $S \to_C T$ (and hence, not $S \to_C x$).

In each of these cases, $P[S \to T|C]$ is equal to p in either of the two models being compared, and this gives us the result. □.

Combining the above three lemmas, we get the following result. Let p_c^{bond} (respectively p_c^{site}) denote the critical value or the percolation for ordinary bond (resp. site) percolation on $\mathbf{Z}^d$.

Theorem 2 *For the epidemic model described above,*

$$P[|C_0| = \infty] = 0 \quad \text{if} \int_0^\infty (1 - e^{-\lambda x}) dF(x) < p_c^{bond};$$
$$P[|C_0| = \infty] > 0 \quad \text{if} \int_0^\infty (1 - e^{-\lambda x}) dF(x) > p_c^{site}.$$

In the case $d = 2$, it is rigorously proved that $p_c^{bond} = 0.5$, amd it is believed on the basis of simulations that p_c^{site} is approximately 0.59. Together these give reasonable bounds on λ_c for $d = 2$.

Notes

The material for Lectures 1 and 3 is adapted from Durrett(1988). Most of the material for Lecture 2 can be found in Bezuidenhout and Grimmett (1990); see also Durrett (1988) and Liggett (1985).

References

[1] Bezuidenhout, C. and Grimmett, G. (1990). The critical contact process dies out. *Ann. Probab.* **18**, 1462-1482.

[2] Cox, J.T. and Durrett, R. (1981) Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **14**, 347-370.

[3] Cox, J.T. and Durrett, R. (1988) Limit theorems for the spread of epidemics and forest fires. *Stoch. Proc. Applns.* **30** (1988) 171-191.

[4] Durrett, R. (1988) *Lecture notes on Particle Systems and Percolation.* Wadsworth & Brooks/Cole, Pacific Grove, California.

[5] Liggett, T.M. (1985) *Interacting Particle Systems.* Springer, New York.

[6] Meester, R. (1996) Lecture notes in percolation (notes for the Workshop in Percolation, Indian Statistical Institute, New Delhi 1996).

Department of Mathematical Sciences
University of Durham
South Road
Durham DH1 3LE
United Kingdom
Mathew.Penrose@durham.ac.uk