Let $d \in \mathbb{N}$. Recall for $x \in \mathbb{R}^d$

$\mu \in \underline{N}$, $\theta_x \mu = \mu(\cdot + x)$

eg $\theta_x \delta_x = \delta_0$

Prop. VI.1 ((8.4) Suppose η is a sta. pt. process in $\mathbb{R}^d$ [i.e. $\theta_x \eta \overset{d}{=} \eta \quad \forall x \in \mathbb{R}^d$]

Then $P[0 < \eta(\mathbb{R}^d) < \infty] = 0$

Proof Let $E = \{0 < \eta(\mathbb{R}^d) < \infty\}$

Suppose $P(E) = \delta > 0$. For $n \in \mathbb{N}$ let $B_n = \{x \in \mathbb{R}^d : |x| \le n\}$.

$$E_n = \{\eta(B_n) = \eta(\mathbb{R}^d)\} \cap E$$

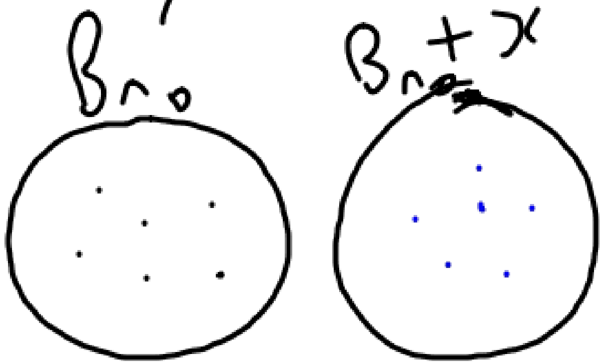
Then $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = E$

$(E_n \uparrow E)$

Choose n_0: $P[E_{n_0}] > \delta/2$

and $x \in \mathbb{R}^d$ with $|x| > 2n_0$

Set $\hat{E}_{n_0} = \{\theta_x \eta(B_{n_0}) = \eta(\mathbb{R}^d)\} \cap E$



By Stationarity

$$P[\hat{E}_{n_0}] = P[E_{n_0}] > \frac{\delta}{2}$$

But $E_{n_0} \cap \hat{E}_{n_0} = \emptyset$ so

$$P[E] \geq P[E_{n_0} \cup \hat{E}_{n_0}] = 2P[E_{n_0}] > \delta. \ \#$$

VI.2 Simple point processes in $\mathbb{R}^d$

A measure $\mu \in \underline{N}(\mathbb{R}^d)$ is said to be:

- <u>locally finite</u> if $\mu(B_n) < \infty \ \forall n \in \mathbb{N}$.
- <u>simple</u> if $\mu(\{x\}) \leq 1 \ \forall x \in \mathbb{R}^d$

Let $\underline{N}_{ls} = \{\mu \in \underline{N}(\mathbb{R}^d) : \mu$ is locally finite and simple$\}$

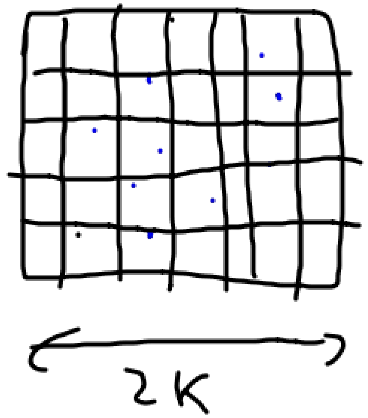
Prop.[A] (6.7) $\underline{N}_{es} \in \mathcal{N}$

proof $\underline{N}_{es} = \bigcap_{K=1}^{\infty} \underline{N}_s^{(K)}$ where

$\underline{N}_s^{(K)} = \{\mu \in \underline{N} : \mu([-K,K]^d) < \infty, \mu \text{ simple}\}$

Fix $K \in \mathbb{N}$. Divide $[-K,K]^d$ into cubes of side $\frac{1}{m}$ $(m \in \mathbb{N})$, denoted $Q_{m,1} \cdots Q_{m,k_m}$

__Claim__ If $\mu \in \underline{N}_s^{(K)}$ then $\exists\, m_0$: $\forall\, m \geq m_0$, $i \leq R_m$ we have $\mu(Q_{m,i}) \leq 1$. (ex.)



$$\underline{N}_s^{(K)} = \bigcup_{m_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} \{\mu \in \underline{N} : \mu((-K,K]^d) < \infty \text{ and } \mu(Q_{m,i}) \leq 1 \ \forall\, i \leq R_m\} \in \mathcal{N}. \ \blacksquare$$

__Defn.__ A point process η in $\mathbb{R}^d$ is said to be locally finite & simple if

$$P[\eta \in \underline{N}_{ls}] = 1$$

Prop B (8.11) Suppose η is a stationary simple locally finite point process in $\mathbb{R}^d$ with $\gamma := \mathbb{E}[\eta([0,1]^d)] \in (0,\infty)$.

Let $r_n > 0$, with $r_n \to 0$ as $n \to \infty$. Set $B_n = \{x \in \mathbb{R}^d : |x| \leq r_n\}$ now. Then

$$\lim_{n\to\infty} \frac{P[\eta(B_n)\geq 1]}{\lambda_d(B_n)} = \lim_{n\to\infty} \frac{P[\eta(B_n)=1]}{\lambda_d(B_n)} = \gamma$$

<u>Proof</u> By Prop. VI.1 A

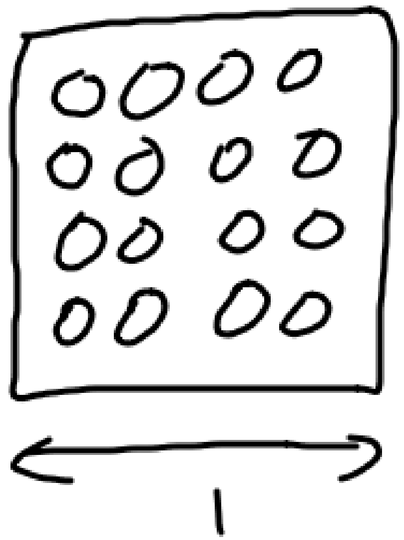
$$\gamma = \frac{\mathbb{E}\,\eta(B_n)}{\lambda_d(B_n)} = \frac{P[\eta(B_n)=1]}{\lambda_d(B_n)} \quad (1)$$

$$+ \frac{\mathbb{E}\left[\eta(B_n)\mathbb{1}_{\{\eta(B_n)\geq 2\}}\right]}{\lambda_d(B_n)} \quad (2)$$

Aim to show

(2) $\to 0$

Let $B_{n,1}, \ldots, B_{n,k_n}$ be disjoint balls of radius r_n contained $[0,1]^d$ with k_n as large as possible



Then $k_n = \Theta(r_n^{-d})$ so $\exists\ \delta > 0$:

$$k_n \geq \delta r_n^{-d} \quad \forall n$$

$$k_n \mathbb{E}\left[\eta(B_n)\mathbb{1}_{\{\eta(B_n)\geq 2\}}\right]$$

$$= \mathbb{E}\sum_{i=1}^{k_n} \eta(B_{n,i})\mathbb{1}_{\{\eta(B_{n,i})\geq 2\}} \quad \text{(by Sta.)}$$

$$\leq \mathbb{E}\int_{[0,1]^d} \mathbb{1}_{\{\eta(B_{2r_n}(x))\geq 2\}}\,\eta(dx)$$

$$[B_a(x) = \{y : |y-x| < a\}]$$

$$\xrightarrow{\text{DOM}} \mathbb{E}\int_{[0,1]^d} \mathbb{1}_{\{\eta(\{x\})\geq 2\}}\,\eta(dx) = 0$$

So $② \le \text{Const.}\, r_n^{-d}\, \mathbb{E}[\eta(B_n) \mathbb{1}_{\eta(B_n)\ge 2}]$

$\le \quad '' \quad R_n \times \cdots$

$\to 0$

So $① \to \gamma$ and

$\frac{P[\eta(B_n) \ge 2]}{\lambda_d(B_n)} \le ② \to 0$ so

combined with $① \to \gamma$

gives $\frac{P[\eta(B_n) \ge 1]}{\lambda_d(B_n)} \to \gamma.$ $\square$

VI-3 The palm distribution (9.2)

<u>Theorem</u> (refined Campbell) (9.1)

Let η be a sta. pt. pr. in $\mathbb{R}^d$ with $\gamma := \mathbb{E}\eta([0,1]^d) \in (0,\infty)$. Then there is a unique probability measure $\mathbb{P}_0$ on $(\underline{N}, \mathcal{N})$ such that $\forall f \in \mathbb{R}_+(\mathbb{R}^d \times \underline{N})$

$$\mathbb{E} \int f(x, \theta_x \eta)\, \eta(dx)$$
$$= \gamma \int \mathbb{E} f(x, \eta_0)\, dx \qquad (*)$$

where η_0 is a pt. pr. with distribution $\mathbb{P}_0$

$\mathbb{P}_0$ is called the Palm distribution of η.

Intuition Set $f(x, \mu) = \mathbb{1}_{\{x \in dy\}}\, g(\mu)$

If η is simple,

$$\text{LHS of } (*) = \mathbb{E}[\eta(dy)\, g(\theta_y \eta)]$$
$$= \underbrace{P[\eta(dy) = 1]}_{\approx \gamma dy}\, \mathbb{E}[g(\theta_y \eta) \mid \eta(dy) = 1]$$

$$\text{RHS of } (*) = \gamma dy\, \mathbb{E}[g(\eta_0)]$$

So $\mathbb{P}_0$ is the distribution of $\theta_y \eta$ given $\eta(dy) = 1$

Proof of Thm

$$\left[\mathbb{E}\int f(x,\theta_x\eta)\,\eta(dx) = \gamma\int \mathbb{E} f(x,\eta_0)\,dx \;(*)\right]$$

Assume $\eta = \sum_{i=1}^{n} \delta_{\xi_i}$ is proper and simple

write $x \in \eta$ if $\eta(\{x\}) = 1$.

For $A \in \mathcal{N}$, $B \in \mathcal{B}^d$ (Borel sets in $\mathbb{R}^d$)

set

$$\mu_A(B) = \mathbb{E} \int 1_B(x) 1_A(\theta_x \eta)\, \eta(dx)$$

$$= \mathbb{E} \sum_{x \in \eta} 1_B(x) 1_A(\theta_x \eta)$$

$\mu_A(\cdot)$ is a measure on $\mathbb{R}^d$. For $B \in \mathcal{B}^d$, $y \in \mathbb{R}^d$, since $x \in B+y \iff x-y \in B$

$$\mu_A(B+y) = \mathbb{E} \sum_{x \in \eta} 1_{B+y}(x) 1_A(\theta_x \eta)$$

$$= \mathbb{E} \sum_{x \in \eta} 1_B(x-y) 1_A(\theta_x \eta)$$

$$\overset{(z=x-y)}{=} \mathbb{E} \sum_{z = \theta_y \eta} 1_B(z) 1_A(\theta_z \theta_y \eta)$$

$$\overset{\eta \overset{d}{=} \theta_y \eta}{=} \mathbb{E} \sum_{z \in \eta} 1_B(z) 1_A(\theta_z \eta)$$

$$= \mu_A(B)$$

μ_A is T.I. so $\exists$
γ_A with $\mu_A = \gamma_A \lambda_d$.

$[\mu_A^{(B)} = \mathbb{E} \int 1_B(x) 1_A(\theta_x \eta)\, \eta(dx)]$

Also $A \mapsto \gamma_A$ is a measure, and

$\gamma_{\underline{N}} = \mu_N([0,1]^d) = \gamma$.

set $\mathbb{P}_0(A) = \frac{\gamma_A}{\gamma}$, $A \in \mathcal{N}$. $\mathbb{P}_0$ is a Prob meas.

For $f(x,\mu) = 1_B(x) 1_A(\mu)$:
$B \in \mathcal{B}^d$, $A \in \mathcal{N}$

$(*) \cdot \mathbb{E} \int 1_B(x) 1_A(\theta_x \eta)\, \eta(dx)$

$= \mu_A(B) = \gamma_A \lambda_d(B) = \gamma\, \mathbb{P}_0(A)\, \lambda_d(B)$

$= \gamma \int \mathbb{E}[1_B(x) 1_A(\eta^0)]\, dx$

$= \gamma \int \mathbb{E}[f(x,\eta^0)]\, dx$

$(*) = \mathbb{E} \int f(x, \theta_x \eta)\, \eta(dx)$

so $(*)$ holds for this f. By monotone class argument, $(*)$ holds $\forall f \in \mathbb{R}_+(\mathbb{R}^d \times \underline{N})$

VII Poisson processes on $\mathbb{R}$. (7)

Suppose $\mathbb{X} = \mathbb{R}^+ = [0, \infty)$.

Given locally finite pt. pr. η, set

$$T_n = \inf\{t : \eta([0,t]) \geq n\}$$

($\inf(\emptyset) := +\infty$). Then T_n is a RV, the <u>n-th arrival time</u>

A PPP on $\mathbb{R}_+$ with intensity $\gamma\lambda$, is called a <u>homogeneous Poisson process</u> (HPP).

VII.1 The interval thm. (7.1)

Theorem (7.2) Suppose η is a locally finite pt process in $\mathbb{R}^+$, $\gamma > 0$.

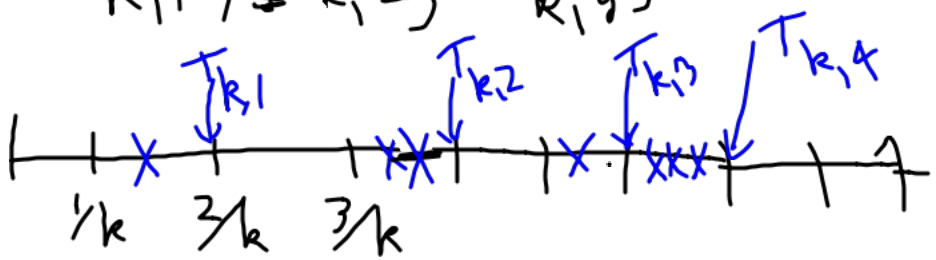
TFAE

(1) η is a HPP with intensity γ

(2) $T_1, T_2 - T_1, \ldots$ are indep. $\exp(\gamma)$.

Proof Assume (1). For $k,n \in \mathbb{N}$ set $I_{k,n} = \left(\frac{n-1}{k}, \frac{n}{k}\right]$. Say $I_{k,n}$ is occupied if $\eta(I_{k,n}) \geq 1$.

Let $T_{k,n}$ be the right endpoint of the n-th occupied interval in $(I_{k,1}, I_{k,2}, I_{k,3}, \dots)$



Put $S_n = T_n - T_{n-1}$, $S_{k,n} = T_{k,n} - T_{k,n-1}$

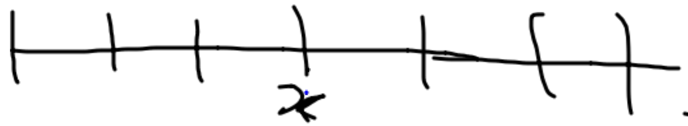
$(T_0 = T_{k,0} = 0)$.

$T_{k,n} \to T_n$ a.s. (as $k \to \infty$) so $S_{k,n} \to S_n$

$(S_{k,n})_{n\geq 1}$ are iid with

$$P[S_{k,1} \geq x] = P[\eta((0,x]) = 0]$$

$$= e^{-\gamma x} \quad \text{if } kx \in \mathbb{N}$$



so $P[S_{k,1} > x] \longrightarrow e^{-\gamma x}$ $\forall x \geq 0$.

$S_{k,n} \xrightarrow{a.s.} S_n$ $(k \to \infty)$ so $S_n \sim \exp(\gamma)$

$S_{k,n} \xrightarrow{\mathcal{D}} \exp(\gamma)$ $(k \to \infty)$

$(S_{k,1}, S_{k,2}, \dots)$ are indep

$S_{k,n} \xrightarrow{a.s.} S_n$ so $S_1, S_2, \dots$ are indep.
i.e (2)

(2) $\Rightarrow$ (1) e.x.