III.1 Defn

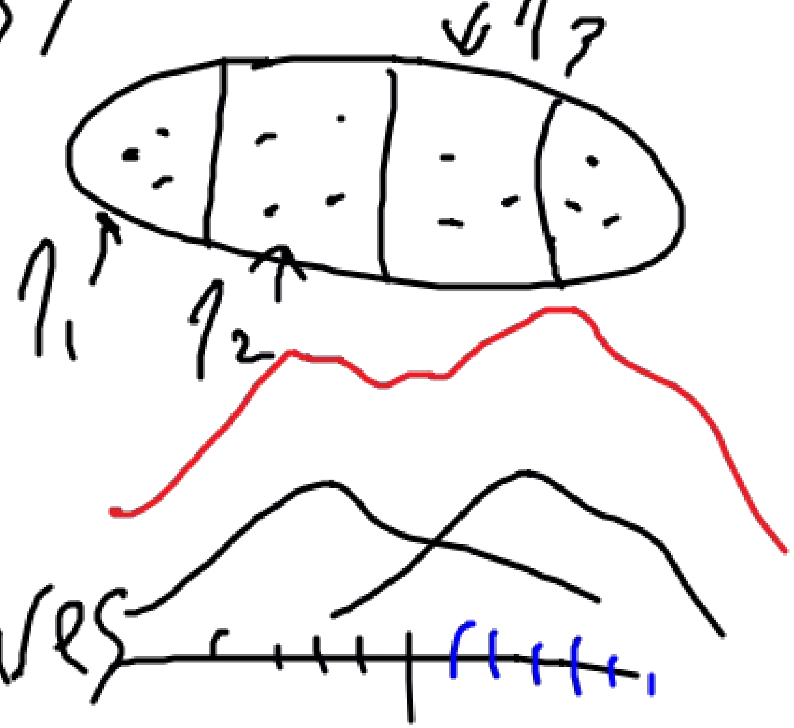
$(\mathbb{X}, \mathcal{X}, \lambda)$ s-finite

η is ppp (λ) on $\mathbb{X}$:

- $\eta(A) \sim Po(\lambda(A))$ $\quad \forall A \in \mathcal{X}$
- $\eta(A_1) \dots \eta(A_n)$ indep if $A_i \cap A_j = \emptyset, i \neq j$

Theorem III.1 (3.3)

(Superposition)



Suppose $\lambda_1, \lambda_2, \dots$ are σ-finite measures on $(\mathbb{X}, \mathcal{X})$ and $\eta_1, \eta_2, \eta_3, \dots$ are indep PPPs on $\mathbb{X}$ with mean measures $\lambda_1, \lambda_2, \dots$ resp.

Set $\eta = \sum_{i=1}^{\infty} \eta_i$ i.e.

$$\eta(A) = \sum_{i=1}^{\infty} \eta_i(A), \quad A \in \mathcal{X}$$

Then η is a PPP with mean measure $\sum_{i=1}^{\infty} \lambda_i$ (which is σ-finite)

$$= \lambda$$

<u>Proof</u> Set $\xi_n = \sum_{i=1}^{n} \eta_i$. Let $A \in \mathcal{H}$. Then $\xi_n(A) \sim Po\left(\sum_{i=1}^{n} \lambda_i(A)\right)$ by Prop. I.1 (& induction)

$\xi_n(A) \to \eta(A)$ a.s. so in distribution

so for $k \in \mathbb{N}_0$,

$$P[\eta(A) = k] = \lim_{n\to\infty} P[\xi_n(A) = k]$$
$$= \lim_{n\to\infty} Po\left(\sum_{i=1}^{n} \lambda_i(A); k\right)$$
$$= Po(\lambda(A); k), \text{ since}$$

$e^{-\gamma} \gamma^k / k!$ is cts. in γ.

Also, if $A_1, A_2, \cdots A_m$ are pairwise disjoint, then

$$\eta_1(A_1), \eta_2(A_1), \cdots$$
$$\eta_1(A_2), \eta_2(A_2), \cdots$$
$$\vdots$$
$$\eta_1(A_m), \eta_2(A_m), \cdots$$

are mutually indep., so

$$\underbrace{\eta(A_1)}_{\substack{\| \\ \sum_{i=1}^{\infty} \eta_i(A_1)}}, \eta(A_2), \cdots \eta(A_m)$$

are indep. $\square$

III.2 Existence of PPPs

Proposition (3.5) Suppose $(\mathcal{X}, \mathcal{H}, \lambda)$ is a meas. space with $\lambda(\mathcal{X}) < \infty$. There exists a proper Poisson pt. process on $\mathcal{X}$ with intensity measure λ.

..

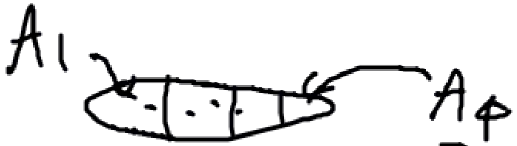
Proof On a suitable $(\Omega, \mathcal{F}, \mathbb{P})$ we can arrange to have

(i) a R.V. $K \sim Po(\lambda)$

$\left[\lambda' = \lambda(\mathbb{X})\right]$

(ii) a sequence $(\xi_1, \xi_2, \dots)$ of indep. random elements of $\mathbb{X}$ with common distribution

$$\mathbb{Q}(\cdot) = \frac{\lambda(\cdot)}{\lambda(\mathbb{X})} \quad (\text{indep. of } K)$$

Set $\eta = \sum_{i=1}^{K} \delta_{\xi_i}$ 

(mixed binomial rep. of a PPP)

η is a proper pt process by def.

For $A_1, A_2, \dots A_n \in \mathcal{X}$ partitioning $\mathbb{X}$

Setting $p_i = Q(A_i)$, $Y_i = \eta(A_i)$:

$K \sim Po(\lambda(\mathbb{X}))$ and for $k \in \mathbb{N}_0$,

$\mathcal{L}(Y_1, \dots, Y_n \mid K = k) \sim \text{mult.}(k; p_1, \dots, p_n)$

By the extension of Prop. I.2,

$Y_i \sim Po(\lambda(\mathbb{X}) \times p_i) = Po(\lambda(A_i))$, and

$Y_1, \dots Y_n$ are indep.

So η is a PPP (λ). ∎

Theorem (3.6) Let $(\mathbb{X}, \mathcal{X}, \lambda)$ be an s-finite. Then there exists a proper PPP on $\mathbb{X}$ with mean measure λ.

proof WLOG $\lambda(\mathbb{X}) = \infty$. Choose measures $\lambda_1, \lambda_2, \dots$ on $(\mathbb{X}, \mathcal{X})$ with $\lambda_i(\mathbb{X}) < \infty$ and $\lambda = \sum_{i=1}^{\infty} \lambda_i$

On a suitable prob. space, assume we have indep. RVs $K_1, K_2, \cdots$ and ξ_{ij} $(i,j \in \mathbb{N})$ with $K_i \sim Po(\lambda_i(\mathbb{X}))$ ($\mathbb{N}_0$-valued) and ξ_{ij} $\mathbb{X}$-valued with distribution

$$Q_i = \frac{\lambda_i(\cdot)}{\lambda_i(\mathbb{X})}.$$

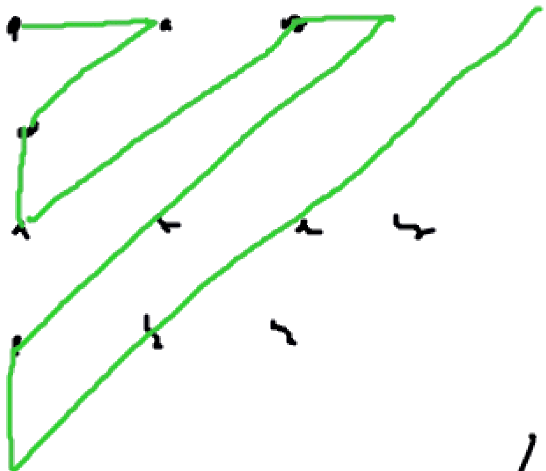
For $i \in \mathbb{N}$, set

$$\eta_i = \sum_{j=1}^{K_i} \delta_{\xi_{ij}} \qquad (K_i \sim Po(\lambda_i(\mathbb{X})))$$

By the lemma, η_i is a PPP with mean measure λ_i. Also $\eta_1, \eta_2, \cdots$ are indep. By Thm. III.1 (Superposition), setting $\eta = \sum_{i=1}^{\infty} \eta_i$, η is a PPP with mean measure $\sum_{i=1}^{\infty} \lambda_i = \lambda$. Also,

$\eta = \sum_{i=1}^{\infty} \sum_{j=1}^{\kappa_i} \delta_{\xi_{ij}}$. Can rewrite

as $\eta = \sum_{k=1}^{\infty} \delta_{\psi_k}$, e.g.



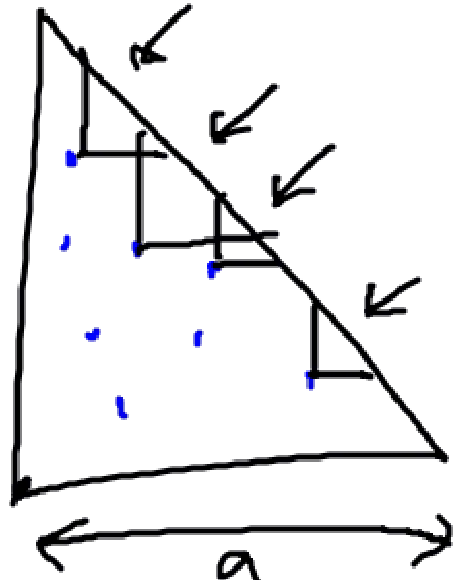
$\kappa_1 = 3$
$\kappa_2 = 1$
$\kappa_3 = 4$
$\kappa_4 = 3$

$(\psi_1, \psi_2, \psi_3 \dots) = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{31}, \xi_{13}, \xi_{32} \dots)$

So η is proper. $\square$

IV The Mecke equation (4)

IV.0 Motivation

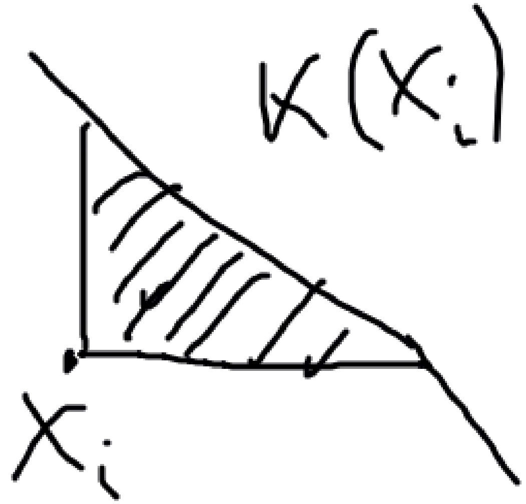
Let $a>0$, $T =$ 

Let $\eta = \sum_i \delta_{X_i}$ is a PPP in T, mean measure $\lambda =$ Lebesgue

let $Z = \sum_i I_i$

$I_i = 1\{\eta(K(X_i)) = 0\}$

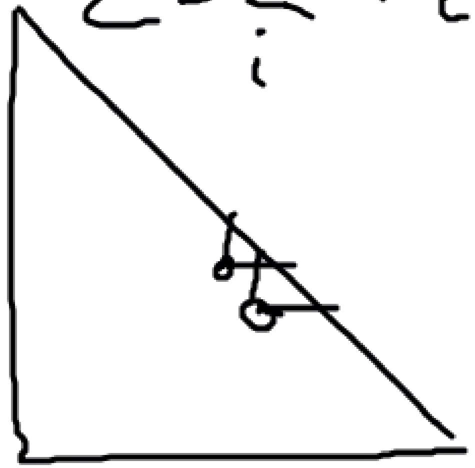
("X_i maximal")



Find $\mathbb{E}[Z]$, $\mathbb{E}[Z^2]$

$Z=\sum_i I_i$

$$Z = \sum_{dx \in T} 1_{\{\eta(dx)=1\}} \times 1_{\{\eta(K(x))=0\}}$$



$$E(Z) = \int_T e^{-\lambda(K(x))} \, dx$$

$Z=\sum_i I_i$

$$Z^2 - Z = \sum_{i \neq j}\sum I_i I_j$$

$$E \sum_{i \neq j}\sum I_i I_j = \int_T \int_T e^{-\lambda(K(x) \cup K(y))} \, dx\, dy$$

IV.1 The univariate Mecke eqn

(4.1) **Theorem (4.1).** Suppose $(\mathbb{X}, \mathcal{X}, \lambda)$ is an σ-finite measure space and η is a pt. process on $\mathbb{X}$ with

$$(*) \quad \mathbb{E} \int_{\mathbb{X}} f(x,\eta)\, \eta(dx) = \int_{\mathbb{X}} \mathbb{E} f(x, \eta + \delta_x)\, \lambda(dx)$$

$\forall f \in \mathbb{R}_+(\mathbb{X} \times \mathbf{N})$. Then η is a PPP on $\mathbb{X}$ with mean measure λ.

[$(*)$ is the univariate Mecke eq.]

[we'll prove a converse later]

Proof Let $A_1, \cdots, A_m \in \mathfrak{X}$ be disjoint with $\lambda(A_i) < \infty$. Let $k_1, \cdots, k_m \in \mathbb{N}_0$. Let

$$f(x,\eta) = \mathbb{1}\{x \in A_m, \eta(A_1) = k_1, \dots \eta(A_m) = k_m\}$$

Then $\mathbb{E} \int_{\mathbb{X}} f(x,\eta)\, \eta(dx)$

$$= \mathbb{E}\left[\prod_{i=1}^{m} \mathbb{1}\{\eta(A_i) = k_i\} \int_{\mathbb{X}} \mathbb{1}_{A_m}\, d\eta\right]$$

$$= k_m\, \mathbb{P}[\eta(A_1) = k_1, \dots, \eta(A_m) = k_m]$$

and for $x \in \mathbb{X}$

$$\mathbb{E} f(x, \delta_x + \eta) = \mathbb{1}_{A_m}(x)\, \mathbb{P}[\eta(A_1) = k_1, \eta(A_2) = k_2, \dots \eta(A_m) = k_m - 1]$$

So by $(*)$

$k_m P[\eta(A_1)=k_1, \cdots \eta(A_m)=k_m]$

$= \lambda(A_m) P\left[\bigcap_{i=1}^{m-1}\{\eta(A_i)=k_i\} \cap \{\eta(A_m)=k_m-1\}\right]$

Set $\pi(k) = P\left[\eta(A_m)=k \,\middle|\, \bigcap_{i=1}^{m-1}\{\eta(A_i)=k_i\}\right]$

$\pi(k) = \frac{\lambda(A_m)}{k}\pi(k-1)$ $(k=k_m)$

$\pi(n) = \pi(0)\frac{\pi(1)}{\pi(0)}\cdots\frac{\pi(n)}{\pi(n-1)} = \frac{\pi(0)\lambda(A_m)^n}{n!}$

So $\pi(n) = P_0(\lambda(A_m); n)$

doesn't depend on $k_1 \cdots k_{m-1}$

so $\eta(A_m) \sim P_0(\lambda(A_m))$ indep. of $\eta(A_1), \cdots \eta(A_{m-1})$

$\eta(A_i)=k_i\}]$

By induction on m,

$\eta(A_1), \cdots \eta(A_m)$ are indep.

If $\lambda(A) = \infty$ then still

$k P[\eta(A) = k] = \lambda(A) P[\eta(A) = k-1]$

So $P[\eta(A) = k-1] = 0 \quad \forall k \in \mathbb{N}$

So $P[\eta(A) = \infty] = 1$

So η is a PPP with mean measure λ. $\square$