MA30253: Continuum Mechanics
Tutorial Sheet 7

Suppose throughout this sheet that $W(z) = u(x,y) - iv(x,y)$, $z = x + iy$, is the complex velocity corresponding to a steady 2D incompressible irrotational flow (see lecture notes).

Let $\Phi(z) = \phi(x,y) + i\psi(x,y)$ denote a corresponding complex potential (so that $\Phi'(z) = W(z)$ on its domain of definition).

1. Show that ϕ and ψ are harmonic functions on their domain of definition. If the velocity potential ϕ corresponds to a steady 2D solution of the Euler equations with body force per unit mass $\mathbf{b} = -\nabla\chi$, show that the corresponding pressure is given by

$$-P = \rho_0 \left(\frac{1}{2} |\nabla\phi|^2 + \chi \right).$$

2. Show that the complex potential $\Phi(z) = U_0 z \exp(-i\alpha)$, with $U_0 > 0$, represents a uniform flow at angle $\alpha > 0$ to the $x-$axis.

3. Show that the complex potential

$$\Phi(z) = \alpha \, \mathrm{Log}(z),$$

defined on the cut plane $\mathbb{C} \backslash \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$, where $\alpha > 0$ is a constant and $\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}\, z$ is the complex logarithm, corresponds to a fluid source at the origin.

4. Show that the complex potential (defined on the cut plane)

$$\Phi(z) = -i\frac{\gamma_0}{2\pi}\mathrm{Log}(z),$$

where $\gamma_0 > 0$ is a constant, represents a flow with a point vortex at the origin with circulation γ_0 around it.

5. Show that $\Phi(z) = U_0 \left(z + \frac{a^2}{z} \right)$ represents the flow past a disc of radius $a > 0$, tending to a uniform flow at infinity of magnitude $U_0 > 0$ parallel to the $x-$axis.

6. (Milne-Thomson Circle Theorem.) Suppose that the complex velocity potential $\Phi(z)$ has all its singularities in the region $|z| > a$. Show that

$$\tilde{\Phi}(z) = \Phi(z) + \overline{\Phi\left(\frac{a^2}{\bar{z}}\right)},$$

where the overbar denotes complex conjugation, defines an analytic function that has the same singularities as Φ in the region $|z| > a$, and has $|z| = a$ as a streamline.

7. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an analytic function and that $f'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. Let $\gamma^{(i)}(s)$, $\gamma^{(i)} : [-a, a] \to \mathbb{C}$, $i = 1, 2$, be two C^1 curves intersecting at z_0 with $\gamma^{(1)}(0) =$

$\gamma^{(2)}(0) = z_0$. Show that the image curves $f(\gamma^{(i)}(s))$, $s \in [-a, a]$, $i = 1, 2$, intersect at $f(z_0)$ at the same angle (the angle between two curves at a point of intersection is the angle between their tangent vectors at that point).