MA30253: Continuum Mechanics
Tutorial Sheet 5

1. Le $\boldsymbol{t} \in \mathbb{R}^3$ be a unit vector. Show that if $\mathbf{x} \in \mathbb{R}^3$, then

$$\boldsymbol{t} \times (\mathbf{x} \times \boldsymbol{t})$$

is the component of $\mathbf{x}$ orthogonal to $\mathbf{t}$. Show that it is equal to

$$(I - \mathbf{t} \otimes \mathbf{t})\mathbf{x}.$$

2. Prove the identity

$$(\mathbf{v}.\nabla)\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \frac{1}{2}\nabla||\mathbf{v}||^2,$$

where $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{v}(\mathbf{x}, t)$ is the vorticity.

3. The Euler equations for flow of an inviscid, incompressible fluid are

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}(\mathbf{x}, t) = -\nabla\left(\frac{P(\mathbf{x}, t)}{\rho_0} + \chi(\mathbf{x}, t)\right), \tag{1}$$

where P is the pressure, $\mathbf{b} = -\nabla\chi$ is the body-force per unit mass, $\mathbf{v}(\mathbf{x}, t)$ is the (spatial) velocity, ρ_0 the constant density, together with the incompressibility condition

$$\nabla.\mathbf{v}(\mathbf{x}, t) = 0.$$

(i) Show, using Q2 and by taking the curl of the equations (1), that the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{\partial \boldsymbol{\omega}(\mathbf{x}, t)}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega}(\mathbf{x}, t) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}(\mathbf{x}, t) = 0$$

(you may find the result from sheet 2 Q2 helpful). Deduce that $\boldsymbol{\omega}$ satisfies

$$\frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = \Gamma\boldsymbol{\omega},$$

where $\frac{D}{Dt}$ represents the material time derivative and Γ is the (spatial) velocity gradient.

(ii) Suppose that $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is a steady solution of (1) and show that $\mathbf{v}$ satisfies

$$\boldsymbol{\omega} \times \mathbf{v} = -\nabla H,$$

where

$$H = \frac{P}{\rho_0} + \chi + \frac{1}{2}||\mathbf{v}||^2.$$

Deduce Bernoulli's Theorem that H is constant along streamlines of the flow.