MA30253: Continuum Mechanics
Tutorial Sheet 4

1. Consider the steady flows with the following (spatial) velocity fields:

(i) $\mathbf{v}(\mathbf{x}) = \Gamma(-x_2, x_1, 0)^T$ on $\mathbb{R}^3$, where $\Gamma > 0$ is a constant,

(ii) $\mathbf{v}(\mathbf{x}) = (x_1, x_2, 0)^T$ on $\mathbb{R}^3$,

(iii) $\mathbf{v}(\mathbf{x}) = \frac{1}{(x_1^2+x_2^2)}(-x_2, x_1, 0)^T$ on $\mathbb{R}^3 \backslash \{\mathbf{x} \,|\, x_1 = x_2 = 0\}$.

Calculate the associated vorticity and the circulation $\oint_C \mathbf{v}.d\mathbf{x}$, where C is the unit circle in the $x_3 = 0$ plane (oriented anticlockwise). Briefly explain your answers and describe the corresponding flows.

2. [Transport Theorem 2]

Let $\boldsymbol{\phi} : \Omega \times [0,T] \to \mathbb{R}^3$ be a motion of a body occupying the initial configuration Ω at $t = 0$. Let $f(\mathbf{x}, t)$ be any smooth function defined for $\mathbf{x} \in \Omega_t = \boldsymbol{\phi}(\Omega, t)$.

Use Reynolds Transport Theorem to prove the second version of the transport theorem:

$$\frac{d}{dt}\int_{\Omega_t} \rho(\mathbf{x}, t)\, f(\mathbf{x}, t)\; dV_{\mathbf{x}} = \int_{\Omega_t} \rho(\mathbf{x}, t)\frac{D}{Dt} f(\mathbf{x}, t)\; dV_{\mathbf{x}},$$

where $\rho(\mathbf{x}, t)$ is the density at time t at $\mathbf{x} \in \Omega_t$. (Hint: use the continuity equation.)

3. [Differentiating contour integrals]

Let $\boldsymbol{\phi} : \Omega \times [0,T] \to \mathbb{R}^3$ be a motion of a body occupying the initial configuration Ω at $t = 0$. Let the contour $C_0 \subset \Omega$ be parametrised as $\mathbf{X}(\lambda)$, $\lambda \in [0,1]$. Let $C_t = \boldsymbol{\phi}(C_0, t)$ denote the image of the contour under the motion at time t. Then C_t is parametrised as $\boldsymbol{\phi}(\mathbf{X}(\lambda), t)$, $\lambda \in [0,1]$. Let $\mathbf{f}(\mathbf{x}, t)$ be any smooth vector field defined for $\mathbf{x} \in \Omega_t = \boldsymbol{\phi}(\Omega, t)$. Then, by the chain rule and the definition of a contour integral,

$$\oint_{C_t} \mathbf{f}.d\mathbf{x} = \int_0^1 \mathbf{f}\left(\boldsymbol{\phi}(\mathbf{X}(\lambda), t), t\right) . \left(F(\mathbf{X}(\lambda), t)\frac{d\mathbf{X}(\lambda)}{d\lambda}\right)\, d\lambda,$$

where $F = (F_{ij})$,

$$F_{ij}(\mathbf{X}(\lambda), t) = \frac{\partial \phi_i(\mathbf{X}(\lambda), t)}{\partial X_j}$$

are the components of the deformation gradient. Use the above expression to prove that

$$\frac{d}{dt}\oint_{C_t} \mathbf{f}.d\mathbf{x} = \oint_{C_t} \left(\frac{D\mathbf{f}}{Dt} + \Gamma^T \mathbf{f}\right).d\mathbf{x}, \tag{1}$$

where Γ denotes the spatial velocity gradient.

In the case that $\mathbf{f} = \mathbf{v}$ (the spatial velocity), show that the circulation around C_t satisfies

$$\frac{d}{dt}\oint_{C_t} \mathbf{v}.d\mathbf{x} = \oint_{C_t} \left(\frac{D\mathbf{v}}{Dt}\right).d\mathbf{x}.$$