MA30253: Continuum Mechanics
Tutorial Sheet 3

1. Show that $\frac{\partial(\det \mathbf{F})}{\partial F_{i\alpha}} = (\mathrm{Cof}\mathbf{F})_{i\alpha}$ on $M^{3\times 3}$, where Cof$\mathbf{F}$ is the cofactor matrix[1] of $\mathbf{F}$.

2. Show that if $\phi(\mathbf{X}, t)$ is a motion and $\mathbf{F} = \left(\frac{\partial \phi_i}{\partial \mathbf{x}_\alpha}\right)$ is the deformation gradient tensor, then

$$\frac{d}{dt}(\det \mathbf{F}(\mathbf{X}, t)) = (\nabla.\mathbf{v}(\mathbf{x}, t))(\det \mathbf{F}(\mathbf{X}, t)), ,$$

where $\mathbf{x} = \phi(\mathbf{X}, t)$.

3. A motion $\phi : \Omega \times [0, T] \to \mathbb{R}^3$ is *isochoric* (volume preserving) if $\det(D\phi(\mathbf{X}, \mathbf{t})) \equiv \mathbf{1}$. Show that the corresponding velocity $\mathbf{v}(\mathbf{x}, t)$ and density[2] $\rho(\mathbf{x}, t)$ then satisfy

(i) $\nabla.\mathbf{v} = \frac{\partial v_i(\mathbf{x},t)}{\partial x_i} = 0$,

(ii) $\frac{D\rho(\mathbf{x},t)}{Dt} = 0$,

(iii) $\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0$,

for $\mathbf{x} \in \Omega_t$, $t \in [0, T]$.

(In fact, all these notions can be shown to be equivalent for sufficiently regular motions.)

4. Show that the spin tensor $\mathbf{W}$ (i.e., the skew symmetric part of the spatial velocity gradient) satisfies

$$\omega(\mathbf{x}, t) \times \mathbf{a} = 2\mathbf{W}\mathbf{a} \quad \forall \mathbf{a} \in \mathbb{R}^3,$$

where $\boldsymbol{\omega} = (\omega_i)$, $\omega_i = \epsilon_{ipq} W_{qp}$, and that $\boldsymbol{\omega} = \nabla \times \mathbf{v}(\mathbf{x}, t)$ (i.e., the vorticity of the flow).

5. Suppose that $f \in C(\Omega)$, that $D_\epsilon(\mathbf{x}_0)$ are subdomains containing $\mathbf{x}_0$ with diameter$(D_\epsilon(\mathbf{x}_0)) \to 0$ as $\epsilon \to 0$. Show that

$$\frac{\int_{D_\epsilon(\mathbf{x}_0)} f(\mathbf{x})\, dV_{\mathbf{x}}}{\mathrm{Vol}(D_\epsilon(\mathbf{x}_0))} \to f(\mathbf{x}_0) \quad \text{as } \epsilon \to 0.$$

Similarly, suppose that $S_\epsilon(\mathbf{x}_0) \subset \Omega$ is a family of surfaces enclosing $\mathbf{x}_0$, and that diameter$(S_\epsilon) \to 0$ as $\epsilon \to 0$. Show that

$$\frac{\int_{S_\epsilon(\mathbf{x}_0)} f(\mathbf{x})\, dA_{\mathbf{x}}}{\mathrm{Area}(S_\epsilon(\mathbf{x}_0))} \to f(\mathbf{x}_0) \text{ as } \epsilon \to 0.$$

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[1] So that $(\mathrm{Cof}\mathbf{F})^T \mathbf{F} = \mathbf{F}(\mathrm{Cof}\mathbf{F})^T = (\det \mathbf{F}) I_n$.

[2] The density $\rho(\mathbf{x}, t)$ at $\mathbf{x} \in \Omega_t$ is related to the density $\rho_0(\mathbf{X})$ at $\mathbf{X} \in \Omega$ by $\rho(\mathbf{x}, t) = \frac{\rho_0(\mathbf{X})}{\det D\phi(\mathbf{X},t)}$, where $\mathbf{x} = \phi(\mathbf{X}, t)$. (This follows by conservation of mass and the change of variables formula for multiple integrals.)