MA30253: Continuum Mechanics
Tutorial Sheet 2

1. Let $f : \Omega \to \mathbf{R}$ be a scalar field, $f = f(X_1, X_2, X_3)$, and let $\mathbf{\Phi} : \Omega \to \mathbb{R}^3$, $\mathbf{\Phi} = \mathbf{\Phi}(X_1, X_2, X_3)$, be a vector field. Show that

(i) $\nabla \times (\nabla f) = \mathbf{0}$, (ii) $\nabla \cdot (\nabla \times \mathbf{\Phi}) = 0$.

2. Prove the following identity

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g} - \mathbf{g}(\nabla \cdot \mathbf{f}) + \mathbf{f}(\nabla \cdot \mathbf{g}),$$

where the vectors $\mathbf{f}, \mathbf{g}$ have components f_i and g_i which are functions of X_1, X_2, X_3 and we have used the operator notation $(\mathbf{g} \cdot \nabla) = g_i \frac{\partial}{\partial X_i}$, and $\nabla \cdot \mathbf{g} = \frac{\partial g_i}{\partial X_i}$ is the divergence of $\mathbf{g}$ (similarly for $\mathbf{f}$).

3. (In this question, $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{x} = (x_1,\ x_2,\ x_3)$ denote material (Lagrangian) and spatial (Eulerian) coordinates respectively.)

A motion $\phi(\mathbf{X}, t) = (\phi_1(\mathbf{X}, t), \phi_2(\mathbf{X}, t), \phi_3(\mathbf{X}, t))$ of a continuous body is given by

$$\phi_1 = X_1 \cos(\omega t) - X_2 \sin(\omega t),\ \ \phi_2 = X_1 \sin(\omega t) + X_2 \cos(\omega t),\ \ \phi_3 = X_3 + t,$$

where $\omega > 0$ is a constant.

Determine the velocity of a material point under this motion and give both the spatial and material representations of the velocity. Find the material rate of change of the following quantities: (a) $f(\mathbf{x}, t) = x_1 - t + x_2$, (b) $f(\mathbf{x}, t) = (x_1)^2 + (x_2)^2$, expressing your answers in Eulerian coordinates.

Describe this motion in physical terms.

4. Find the particle paths, the velocity gradient, rate of stretch and spin tensors corresponding to the following (spatial) velocity fields $\mathbf{v} = (v_1, v_2, v_3)^T$:

(i) $v_1(\mathbf{x}, t) = x_2,\ v_2(\mathbf{x}, t) = x_1,\ v_3(\mathbf{x}, t) = 0,$

(ii) $v_1(\mathbf{x}, t) = 0,\ \ v_2(\mathbf{x}, t) = -x_3,\ v_3(\mathbf{x}, t) = x_2,$

where $\mathbf{x} = (x_1, x_2, x_3)^T$.

5. Find the right polar decomposition of the matrix

$$\begin{pmatrix} 2 & -3 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$