MA30253: Continuum Mechanics
Tutorial Sheet 1

1. Determine which of the following expressions is valid when using the Einstein summation convention:

(i) $a_{ij}c_{ij} = a_{ij}c_{jj}$, (ii) $a_{ij} = b_i b_j$, (iii) $a_{ii} = b_{jj}$, (iv) $\frac{\partial a_{ij}}{\partial x_k} = c_{ik}b_j$, (v) $c_{ij} = \frac{\partial a_i}{\partial x_i} + \frac{\partial b_j}{\partial x_j}$.

2. (i) Write the matrix equation

$$CD^T = A(I - BB^T)C^T$$

using index notation.

(ii) Which of the following expressions is equivalent to $v_i = b_{ji}\tilde{v}_j$: (a) $v_k = b_{mk}\tilde{v}_m$, (b) $v_p = b_{pq}\tilde{v}_q$, (c) $v_m = \tilde{v}_n b_{nm}$?

3. Write down the Euler equations for the flow of an inviscid fluid together with the condition of incompressibility using index notation.

4. If $S = (s_{ij})$ is a symmetric matrix and $A = (a_{ij})$ is a skew-symmetric matrix, show that $a_{ij}s_{ij} = 0$.

5. Prove the following identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \ .$$

6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$, then the *tensor product* of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \otimes \mathbf{b}$, is the linear transformation from $\mathbb{E}^3$ to $\mathbb{E}^3$ defined by

$$\mathbf{a} \otimes \mathbf{b}\,(\mathbf{v}) = \langle \mathbf{b}, \mathbf{v} \rangle \mathbf{a} \quad \forall \mathbf{v} \in \mathbb{E}^3.$$

Suppose that $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ is a linear transformation and that $\mathrm{A} = (a_{ij}) \in \mathrm{M}^{3\times 3}$ is the matrix representing σ with respect to the basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in both domain and codomain. Show that

$$\sigma = a_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j.$$

Deduce that $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is a basis for the vector space $\mathcal{L}(\mathbb{E}^3, \mathbb{E}^3)$ of linear transformations from $\mathbb{E}^3$ to $\mathbb{E}^3$. What is the dimension of $\mathcal{L}(\mathbb{E}^3, \mathbb{E}^3)$?

7. Using the result from Q5, prove the following formula for the triple vector product of $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{E}^3$

$$\mathbf{X} \times (\mathbf{Y} \times \mathbf{Z}) = \langle \mathbf{X}, \mathbf{Z} \rangle \mathbf{Y} - \langle \mathbf{X}, \mathbf{Y} \rangle \mathbf{Z} \ .$$