MA30253: Continuum Mechanics

Prerequisites:

MA20223 - Vector Calculus and PDEs

MA20219 - Analysis 2B

We will also utilise basic ideas/results from MA20216 (Algebra 2A).

Introduction

This course develops a general theory of continuum mechanics which can be applied to modelling gases, fluids and elastic solids. In a continuum theory, we assume that the material is infinitely finely divisible, ignoring the atomic structure. This turns out to provide realistic models and predictions at length scales much larger than the interatomic distance.

Let $\Omega \subset \mathbb{R}^3$ denote a reference configuration of our continuum and let $\boldsymbol{\phi}(\mathbf{X}, t)$, $\boldsymbol{\phi} : \Omega \times [0, t_0] \to \mathbb{R}^3$, $t_0 > 0$, denote a motion of the continuum. Then the continuum occupies the region $\Omega_t = \boldsymbol{\phi}(\Omega, t) \subset \mathbb{R}^3$ at a particular time t. We ascribe to the continuum a *density* $\rho(\mathbf{x}, t)$ and *velocity* $\mathbf{v}(\mathbf{x}, t) = (v_i(\mathbf{x}, t))$ at each point $\mathbf{x} = (x_i) \in \Omega_t$ of the body. (We will refer to $\mathbf{X}$ as material coordinates and $\mathbf{x}$ as spatial coordinates.)

We postulate that internal forces within the continuum are given by the *stress vector* $\mathbf{t}(\mathbf{x}, \mathbf{n}) = (t_i(\mathbf{x}, \mathbf{n}))$ which gives the force per unit area on a surface through the point $\mathbf{x} \in \Omega_t$ with unit normal $\mathbf{n}$ at $\mathbf{x}$ at a given time t. This is known as the *Cauchy-Euler hypothesis.* We will deduce later in the course a result due to Cauchy that, under suitable assumptions, the dependence on $\mathbf{n}$ is linear and given by $\mathbf{t}(\mathbf{x}, \mathbf{n}) = T(\mathbf{x})\mathbf{n}$, where T is known as the *Cauchy stress tensor* and can be identified with a 3×3 matrix (T_{ij}) in any given cartesian (i.e., right-handed orthonormal system) coordinate system.

We postulate *conservation of mass*, from which it will follow that the density $\rho(\mathbf{x}, t)$ satisfies the scalar PDE

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0. \tag{0.1}$$

Similarly, postulating *balance of angular momentum and linear momentum* will imply that the stress tensor is symmetric and satisfies the system of partial differential equations

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = Div\, T + \rho \mathbf{F} \quad , \tag{0.2}$$

where $\mathbf{F}(\mathbf{x}, t) = (F_i(\mathbf{x}, t))$ is the body force per unit mass and $\mathbf{v}.\nabla$ denotes the differential operator $\sum_{k=1}^{3} v_k(\mathbf{x}, t)\frac{\partial}{\partial x_k}$. In the above notation, $Div\, T$ denotes the vector obtained by taking the divergence of each row (considered as a vector) of the Cauchy stress tensor. Hence the system (0.2) consists of three PDEs

$$\rho(\mathbf{x}, t)\left(\frac{\partial v_i(\mathbf{x}, t)}{\partial t} + \sum_{k=1}^{3} v_k(\mathbf{x}, t)\frac{\partial}{\partial x_k} v_i(\mathbf{x}, t)\right) = \sum_{k=1}^{3} \frac{\partial T_{ik}}{\partial x_k}(\mathbf{x}, t) + \rho F_i(\mathbf{x}, t) \ , \quad i = 1, 2, 3.$$

To complete the model of a particular continuum, we specify a *constitutive law* which gives the form of the Cauchy stress tensor (and relates it to the other variables in the problem).

Example

For example, in the case of an incompressible, ideal fluid, we set $\rho = \rho_0$ (a constant) and

$$T(\mathbf{x}, t) = -P(\mathbf{x}, t)\, I, \tag{0.3}$$

where P is a scalar function called the pressure and I denotes the 3×3 identity matrix.[1] In this case, we obtain the **Euler Equations** governing ideal, inviscid flow:

$$\rho_0\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = -\nabla P + \rho_0 \mathbf{F} \quad . \tag{0.4}$$

The conservation of mass equation (0.1) then reduces to the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \ \text{ in } \Omega. \tag{0.5}$$

Vorticity

An important concept in the study of fluid flows is the *vorticity* $\boldsymbol{\omega}(\mathbf{x}, t) = (\omega_i(\mathbf{x}, t))$, defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{0.6}$$

This is a measure of the rotation inherent in the flow. We say that the flow is *irrotational* if the vorticity (0.6) is identically zero. A flow is said to be *steady* if $\mathbf{v}(\mathbf{x}, t)$ is independent of time, i.e., if $\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0$.

[1] This presumes that $\mathbf{t}(\mathbf{x}, \mathbf{n})$ is parallel to $\mathbf{n}$ which is not a reasonable assumption in the case of viscous fluids such as oil.

Example (steady, irrotational, planar flow)

$$\mathbf{v} = \begin{pmatrix} v_1(x_1,x_2) \\ v_2(x_1,x_2) \\ 0 \end{pmatrix} \Longrightarrow \nabla \times \mathbf{v} = \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}$$

so, from vector analysis, if the flow is irrotational then there exists $\Phi(x_1, x_2)$ such that:

$$\mathbf{v} = \nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x_1} \\ \frac{\partial \Phi}{\partial x_2} \\ 0 \end{pmatrix}.$$

The function Φ is called the *velocity potential*. If the flow is also incompressible, then by (0.5) we have

$$0 = \nabla \cdot \mathbf{v} = \nabla \cdot (\nabla\Phi) = \Delta\Phi,$$

i.e., Φ satisfies Laplace's Equation. In this case, if we further suppose that $\mathbf{F} = \nabla\psi$, then choosing $-P = \frac{1}{2}\rho_0|\mathbf{v}|^2 - \rho_0\psi$ yields a solution of the Euler equations (0.4).

Recall from complex analysis (see MA20219) that the real and imaginary parts of a complex analytic function satisfy Laplace's equation. This suggests that we may be able to apply examples and results from complex analysis to study these flows.

Example

$\Omega = \{\mathbf{x} = (x_1, x_2, x_3) \mid x_2 \geq 0\}$.

Let $\Phi = (x_1)^2 - (x_2)^2$, then $\mathbf{v} = \begin{pmatrix} 2x_1 \\ -2x_2 \\ 0 \end{pmatrix}$ is a solution.

(Note that Φ is the real part of the analytic function $f(z) = z^2,\ z \in \mathbb{C},\ z = x_1 + ix_2$.)

The Euler equations for an ideal (inviscid), incompressible fluid is the main example of a continuum theory which we will study in this course.