

Program Verification in the presence of complex numbers, functions with branch cuts etc.

James Davenport (Bath)

Thanks to Russell Bradford, Acyr Locatelli, Gregory Sankaran and David Wilson (Bath) and Scott McCallum (Macquarie)

28 September 2012

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To this, I wish to add a fourth kind

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(0%!) turns out not to be correct when interpreted, not as abstract algebra, but as the manipulation of functions $\mathbf{R} \rightarrow \mathbf{R}$ or $\mathbf{C} \rightarrow \mathbf{C}$.

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- COMPLEX in Fortran II was the first computer data type not corresponding to a machine one
- Even C99 introduced `_Complex`
- Many examples, notably in fluid mechanics.

Flow in a slotted strip, transformed by

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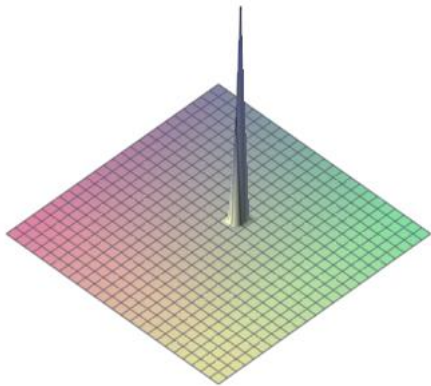
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Or possibly

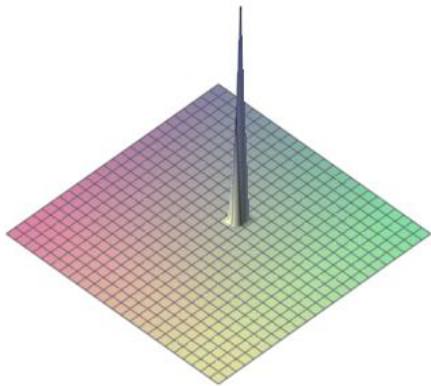
$$w = h(z) := 2 \ln \left(\frac{1}{3} \frac{\sqrt{3z + 12} (\sqrt{z + 3} + \sqrt{z})^2}{2\sqrt{z + 3} + \sqrt{z}} \right)? \quad (3)$$

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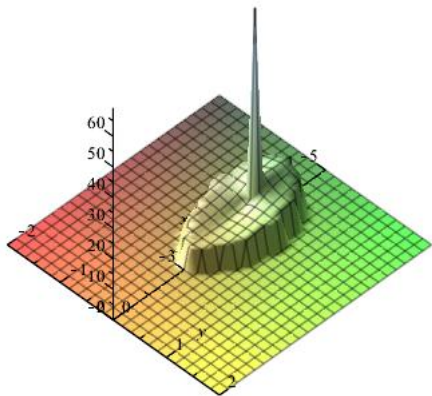


$g - q$ might look OK



“OK apart from a slight glitch.”

But if we look closer



Definitely not OK

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$$\begin{aligned} \frac{d(g-h)}{dz} &= 2 \left(\sqrt{\frac{z}{z+4}} \sqrt{\frac{z+3}{z+4}} z^{3/2} - 2z^{3/2} + 2\sqrt{z+3} \sqrt{\frac{z}{z+4}} \right. \\ &\quad \left. \sqrt{\frac{z+3}{z+4}} z - z\sqrt{z+3} + 4 \sqrt{\frac{z}{z+4}} \sqrt{\frac{z+3}{z+4}} \sqrt{z} + 8\sqrt{z+3} \sqrt{\frac{z}{z+4}} \right. \\ &\quad \left. \sqrt{\frac{z+3}{z+4}} - 6\sqrt{z} \right) \frac{1}{\sqrt{z+3}} \frac{1}{\sqrt{z}} \frac{1}{\sqrt{\frac{z}{z+4}}} \frac{1}{\sqrt{\frac{z+3}{z+4}}} (z+4)^{-2} \left(2\sqrt{z+3} + \sqrt{z} \right)^{-1} \end{aligned}$$

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and it’s a bold person who would say “= 0”

Challenges

Challenge (1)

Demonstrate automatically that g and q are not equal, by producing a z at which they give different results.

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Again, the technology in [BBDP07], implemented in a mixture of Maple and QEPCAD, could in principle do this

Joukowski (a)

Consider the Joukowski map [Hen74, pp. 294–298]:

$$f : z \mapsto \frac{1}{2} \left(z + \frac{1}{z} \right). \quad (4)$$

Lemma

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This is also a function $\mathbf{R}^2 \rightarrow \mathbf{R}^2$:

$$f_R : (x, y) \mapsto \left(\frac{1}{2}x + \frac{1}{2} \frac{x}{x^2 + y^2}, \frac{1}{2}y - \frac{1}{2} \frac{y}{x^2 + y^2} \right) \quad (5)$$

Challenge (3)

Demonstrate automatically that f_R is injective, i.e.

$$\begin{aligned} \forall x_1 \forall x_2 \forall y_1 \forall y_2 & \quad \left(x_1^2 + y_1^2 > 1 \wedge x_2^2 + y_2^2 > 1 \wedge \right. \\ x_1 + \frac{x_1}{x_1^2 + y_1^2} = x_2 + \frac{x_2}{x_2^2 + y_2^2} & \quad \wedge \quad \left. y_1 - \frac{y_1}{x_1^2 + y_1^2} = y_2 - \frac{y_2}{x_2^2 + y_2^2} \right) \\ & \quad \Rightarrow \quad (x_1 = x_2 \wedge y_1 = y_2). \end{aligned} \tag{6}$$

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Challenge (4)

Automate these techniques and transforms.

Joukowski challenge (a2)

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The only challenge might be the choice implicit in the \pm symbol: which do we choose? Unfortunately, the answer is “neither”, or at least “neither, uniformly”.

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$$f_1(\zeta) = \zeta \begin{cases} +\sqrt{\zeta^2 - 1} & \Im(\zeta) > 0 \\ -\sqrt{\zeta^2 - 1} & \Im(\zeta) < 0 \\ +\sqrt{\zeta^2 - 1} & \Im(\zeta) = 0 \wedge \Re(\zeta) > 1 \\ -\sqrt{\zeta^2 - 1} & \Im(\zeta) = 0 \wedge \Re(\zeta) < -1 \end{cases} \quad (7)$$

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Challenge (5)

Derive automatically, and demonstrate the validity of, either (7) or (8). In terms of Maple, this would be ...

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As far as I can tell (supported by the documentation), Maple ignores the “assuming” as it’s on the codomain, not the domain.

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Challenge (6)

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Brown's ideas probably apply here as well, but this is unproven

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Challenge (8)

Demonstrate automatically that this is an inverse to f on $\{z : \Im z > 0\}$.

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A more detailed comparison is given in [Hon91]. Hence there is a need for practical research on low- N Cylindrical Algebraic Decomposition.

Why is this so hard? (1) — CAD continued

While the fundamental branch cut of \log is simple enough, being $\{z = x + iy \mid y = 0 \wedge x < 0\}$, actual branch cuts are messier. Part of the branch cut of (2) is

$$2x^3 + 21x^2 + 72x + 2xy^2 + 5y^2 + 81 = 0 \wedge \text{other conditions, (12)}$$

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whose solution accounts for the curious boundary of the bad region. While there has been some progress in manipulating such images of half-lines (described in Phisanbut's Bath PhD), there is almost certainly more to be done.

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- 1 The statements require the $|\cdot|$ or \Im functions, neither of which are \mathbf{C} analytic functions. Hence some recourse to \mathbf{R} (twice as many variables) seems inevitable (proof?)
- 2 Equations (6) and (9) are the direct translations of the basic definition of injectivity. In practice, certainly if we were looking at functions $\mathbf{R} \rightarrow \mathbf{R}$, we would want to use the fact that the function concerned was continuous.

Why is this so hard? (2) — Injectivity

Lemmas 1, 2 are statements about complex functions of one variable, so why do we need statements about four real variables to prove them? There are three reasons.

- 1 The statements require the $|\cdot|$ or \Im functions, neither of which are \mathbf{C} analytic functions. Hence some recourse to \mathbf{R} (twice as many variables) seems inevitable (proof?)
- 2 Equations (6) and (9) are the direct translations of the basic definition of injectivity. In practice, certainly if we were looking at functions $\mathbf{R} \rightarrow \mathbf{R}$, we would want to use the fact that the function concerned was continuous.

Challenge (9)

Find a better formulation of injectivity questions $\mathbf{R}^N \rightarrow \mathbf{R}^N$, making use of the properties of the functions concerned (certainly continuity, possibly rationality).

Why is this so hard? (2) — Injectivity continued

- 1 While these injectivity equations are statements from the existential theory of the reals, and so the theoretically more efficient algorithms quoted in [Hon91] are in principle applicable, the more modern developments described in [PJ09] do not seem to be directly applicable. However, we can transform them into a disjunction of statements to each of which the Weak Positivstellensatz [PJ09, Theorem 1] is applicable.

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Challenge (10)

Solve these problems using the techniques of [PJ09],

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