$G5 := [a+b+c+d+e, a \cdot b+b \cdot c+c \cdot d+d \cdot e+e \cdot a, a \cdot b \cdot c+b \cdot c \cdot d+c \cdot d \cdot e+d \cdot e \cdot a+e \cdot a \cdot b, a \cdot b \cdot c \cdot d+b \cdot c \cdot d \cdot e+c \cdot d \cdot e \cdot a+d \cdot e \cdot a \cdot b+e \cdot a \cdot b \cdot c, a \cdot b \cdot c \cdot d \cdot e-1];$

$$[a+b+c+d+e, a b+a e+b c+c d+d e, a b c+a b e+a d e+b c d+c d e, a b c d + a b c e+a b d e+a c d e+b c d e, a b c d e-1] \tag{1}$$

$with(Groebner);$

$$[Basis, FGLM, HilbertDimension, HilbertPolynomial, HilbertSeries, Homogenize, InitialForm, InterReduce, IsBasis, IsProper, IsZeroDimensional, LeadingCoefficient, LeadingMonomial, LeadingTerm, MatrixOrder, MaximalIndependentSet, MonomialOrder, MultiplicationMatrix, MultivariateCyclicVector, NormalForm, NormalSet, RationalUnivariateRepresentation, Reduce, RememberBasis, SPolynomial, Solve, SuggestVariableOrder, Support, TestOrder, ToricIdealBasis, TrailingTerm, UnivariatePolynomial, Walk, WeightedDegree] \tag{2}$$

$G5td := Basis(G5, tdeg(a, b, c, d, e));$

$$\begin{aligned}&[a+b+c+d+e, b^2+b d+2 b e-c d+c e+e^2, b c d-2 b d^2-2 b d e+3 b e^2+c^3\\ &+3 c^2 e-c d^2-2 c d e+3 c e^2-d^3-3 d^2 e-2 d e^2+2 e^3, b c^2-b c d+b d e-b e^2\\ &+c^2 d-c^2 e+c d e-2 c e^2+d^2 e+d e^2-e^3, 14 b c d e-b c e^2-27 b d^2 e-10 b d e^2\\ &+24 b e^3+6 c^2 d e+7 c^2 e^2+2 c d^2 e-9 c d e^2+33 c e^3+d^4-15 d^3 e-33 d^2 e^2\\ &-14 d e^3+22 e^4, -2 b c d e-b c e^2+5 b d^2 e+2 b d e^2-4 b e^3-c^2 d e-2 c^2 e^2+c d^3\\ &-7 c e^3+4 d^3 e+7 d^2 e^2+2 d e^3-4 e^4, -5 b c d e+b d^3+10 b d^2 e+2 b d e^2-8 b e^3\\ &-2 c^2 d e-3 c^2 e^2+c d^2 e+2 c d e^2-13 c e^3+6 d^3 e+13 d^2 e^2+4 d e^3-8 e^4, b c d e\\ &-2 b d^2 e-2 b d e^2+3 b e^3+c^2 d^2+2 c d^2 e-2 c d e^2+2 c e^3-d^3 e-2 d^2 e^2-2 d e^3\\ &+2 e^4, b c d^2+b c d e-b c e^2-b d^2 e-b d e^2+b e^3+c^2 d e+c d^2 e+c e^3-d^3 e\\ &-2 d^2 e^2-d e^3+e^4, 2 b c d e^2-b c e^3-2 b d e^3+b e^4+c^2 d e^2+2 c d^2 e^2-c d e^3+c e^4\\ &-d^2 e^3-2 d e^4+e^5-1, b e^5-c e^5-b+c, -20 b c e^4+5 b d e^4-20 c^2 e^4+15 c d^2 e^3\\ &-25 c d e^4-23 c e^5+10 d^3 e^3+30 d^2 e^4-3 d e^5-4 e^6+15 b-7 c+3 d+24 e, b c e^4\\ &+3 b d^2 e^3-b d e^4+c^2 e^4+2 c d e^4-2 c e^5+d^3 e^3-e^6-3 b+2 c-3 e, 2 c d e^5\\ &+8 c e^6+d^2 e^5+d e^6+3 e^7-2 c d-8 c e-d^2-d e-3 e^2, c^2 e^5+3 c e^6+e^7-c^2\\ &-3 c e-e^2, 9 c e^6+5 d^3 e^4+11 d^2 e^5+2 d e^6+3 e^7-10 b c+10 b d-10 c^2-5 c d\\ &-24 c e+4 d^2-2 d e+7 e^2, e^8+42 b c d-76 b c e-165 b d^2+13 b d e+186 b e^2\\ &+21 c^2 d-55 c^2 e+42 c d^2-131 c d e+21 c e^2-55 d^3-21 d^2 e-42 d e^2+219 e^3,\\ &d e^7-110 b c d+29 b c e+52 b d^2-34 b d e+63 b e^2-55 c^2 d-26 c^2 e+60 c d^2\\ &-102 c d e-120 c e^2+39 d^3+120 d^2 e+109 d e^2-26 e^3, c e^7-16 b c d+29 b c e\\ &+63 b d^2-5 b d e-71 b e^2-8 c^2 d+21 c^2 e-16 c d^2+50 c d e-9 c e^2+21 d^3\\ &+8 d^2 e+16 d e^2-84 e^3, d^2 e^6+28 b c d-11 b c e-21 b d^2+9 b d e-5 b e^2+14 c^2 d\\ &+3 c^2 e-12 c d^2+17 c d e+29 c e^2-12 d^3-30 d^2 e-28 d e^2+18 e^3]\end{aligned} \tag{3}$$

 # Total degree bases are generally the cheapest to compute, but may not be very enlightening. We can convert 0-dimensional bases to other orderings with the FGM algorithm

$G5lex := FGLM(G5td, tdeg(a, b, c, d, e), plex(a, b, c, d, e));$

$$[e^{15}+122e^{10}-122e^5-1, -2de^{11}-8e^{12}+55d^2e^5-231de^6-979e^7-55d^2+233de+987e^2, -398de^{11}-1042e^{12}+55d^7+165d^6e+55d^5e^2-48554de^6-127116e^7-55d^2+48787de+128103e^2, e^{11}+55ce^5+143e^6-55c-144e, -442de^{11}-1121e^{12}+440d^6e+1210d^5e^2-275d^3e^4-53911de^6-136763e^7+275cd-275ce+275d^2+53913de+136674e^2, -232de^{12}-568e^{13}+275d^6e^2+550d^5e^3-550d^4e^4-28336de^7-69289e^8+275c^3+550c^2e-550ce^2+550d^2e+28018de^2+69307e^3, e^{11}+55be^5+143e^6-55b-144e, 124de^{11}+346e^{12}-110d^6e-440d^5e^2-275d^4e^3+275d^3e^4+15092de^6+42218e^7+275bd-275be-15106de-42124e^2, 334de^{11}+867e^{12}-330d^6e-1045d^5e^2-275d^4e^3+275d^3e^4+40722de^6+105776e^7+275bc-275be+275c^2+550ce-550d^2-40726de-105873e^2, -566de^{11}-1467e^{12}+550d^6e+1650d^5e^2+275d^4e^3-550d^3e^4-69003de^6-178981e^7+275b^2+825be+275d^2+69019de+179073e^2, a+b+c+d+e] \tag{4}$$

 # Now it's obvious that e is determined (by the degree 15 polynomial
e15 := G5lex[1];

$$e^{15}+122e^{10}-122e^5-1 \tag{5}$$

d2 := collect(G5lex[2], d); # and it looks as if this polynomial determines d

$$(55e^5-55)d^2+(-2e^{11}-231e^6+233e)d-8e^{12}-979e^7+987e^2 \tag{6}$$

 # However, its leading coefficient divides e15, so if e is one of the roots of that e^5-1,
then *this polynomial vanishes (Gianni − Kalkbrener theorem)*

e15v1 $:= e^5-1$; *e15v2 := simplify* $\left(\frac{e15}{e15v1}\right)$;
 # we really need to consider the two cases of e15 separately

$$e^5-1$$

$$e^{10}+123e^5+1 \tag{7}$$

G5lexv1 := Basis([e15v1, op(G5lex)], plex(a, b, c, d, e));

$$[e^5-1, d^7+3d^6e+d^5e^2-d^2-3de-e^2, 8d^6e+22d^5e^2-5d^3e^4+5cd-5ce+5d^2-8de-22e^2, d^6e^2+2d^5e^3-2d^4e^4+c^3+2c^2e-2ce^2+2d^2e-2de^2-2e^3, -2d^6e-8d^5e^2-5d^4e^3+5d^3e^4+5bd-5be+2de+8e^2, -6d^6e-19d^5e^2-5d^4e^3+5d^3e^4+5bc-5be+5c^2+10ce-10d^2+6de+14e^2, 2d^6e+6d^5e^2+d^4e^3-2d^3e^4+b^2+3be+d^2-2de-5e^2, a+b+c+d+e] \tag{8}$$

G5lexv2 := Basis([e15v2, op(G5lex)], plex(a, b, c, d, e));

$$[e^{10}+123e^5+1, -2de^6-8e^7+55d^2-233de-987e^2, e^6+55c+144e, e^6+55b+144e, -2e^6+55a+55d-233e] \tag{9}$$

 # this one is simple: 10 options for e, each of which gives 2 for d, one for c,b,a: 20roots in all.
 # G5lexv1 will prove move complicated
c1 := collect(G5lexv1[3], c);

$$(5d-5e)c+8d^6e+22d^5e^2-5d^3e^4+5d^2-8de-22e^2 \tag{10}$$

 # so d=e is a special case: the rest is left as an exercise!