Mathematical Notation

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The lay person's view

difficult "Pour moi, c'est de l'Algèbre" is the French for "It's all Greek to me"

universal science fiction stories have humans communicating with aliens via mathematics

precise "mathematically precise" is a common phrase

unambiguous follows naturally from precise

The mathematician's view

difficult: Clearly not, and indeed helpful:

[Bou70]: the abuses of language without which any mathematical text threatens to become pedantic and even unreadable.

universal: Pretty much so, though we all have our horror stories

precise: Well, of course, otherwise we wouldn't use it.

unambiguous: follows naturally from precise

If you challenge the mathematician

on unambiguity, say with “is $(1, 2)$ a

permutation (in cycle notation)

an open interval (at least if you’re anglo-saxon)

a (row) vector perhaps you wouldn’t put commas in, but then what is $\binom{3}{2}$?

an ordered pair of integers

. . . ”?

the response is “it’s unambiguous in *my* context”: a tactical retreat to *local unambiguity*

Local unambiguity is **very** local

Consider the following fragments

A $G_i < G \quad \forall i < n$

A' G sub i is a subgroup of G for i less than n

B $R_i < R \quad \forall i < n$

B' R sub i is a subring of R for i less than n

C $k_i < K \quad \forall i < n$

C' k sub i is a subfield of K for i less than n

This causes real problems for my colleague who reads examination papers to blind students

“any identifier is as good as any other”

is what we all preach, but $E = mc^2$ is not $A = \pi r^2$ and

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shows that our practice is rather different.
[Wat08] shows that each top-level MSC has a unique distribution of the first six identifiers (37 [Dynamical Systems] and 58 [Global Analysis] share n, x, i, k, t)

Juxtaposition

It is normal to say that juxtaposition indicates multiplication (MathML's symbol `⁢`) or function application (MathML's `⁡`) [Dav08], but in fact the general rules are more complex, and highly context-sensitive. In general, we can state the observed properties of juxtaposition as

left weight	right weight	meaning	example
normal	normal	lexical	sin
normal	italic	application	$\sin x$
italic	italic	multiplication	xy
		(or `⁣`)	M_{ij}
italic	normal	multiplication	$a \sin x$
digit	digit	lexical	42
		(or `⁣`)	M_{12}
digit	italic	multiplication	$2x$
digit	normal	multiplication	$2 \sin x$

Juxtaposition Table continued

normal	digit	application	$\sin 2$
		(but note the precedence in	$2 \sin 3x$)
italic	digit	error	$x2$
		(but reconsider)	x^2 or x_2?
digit	fraction	addition	$4\frac{1}{2}$
		`&InvisiblePlus;`	
italic	greek	application^{-1}	$a\phi$
		(as in group theory)	i.e. $\phi(a)$
italic	(	unclear	$f(y+z)$ or $x(y+z)$
		what is $f(g+h)$?	

Typography (`\thinspace` etc.) can help, but how many document readers (automatic or untrained human) recognise this?

Can any-one explain satisfactorily why $2 \sin 3x \cos 4x$ means $2 \cdot (\sin(3 \cdot x)) \cdot (\cos(4 \cdot x))$, and not, say, $2 \cdot (\sin(3 \cdot x \cdot \cos(4 \cdot x)))$ or $2 \cdot (\sin 3) \cdot (x \cdot \cos(4 \cdot x))$?

"trigs abhor nesting" isn't sufficient?

Though this is GDML not ICMI

it is worth noting that this overloading of juxtaposition causes real pedagogic problems

> *To illustrate this, I often ask teachers to write $4x$ and $4\frac{1}{2}$. I then ask them what the mathematical operation is between the 4 and the x, which most realize is multiplication. I then ask what the mathematical operation is between the 4 and the $\frac{1}{2}$, which is, of course, addition . . . [Wil11, p. 53].*

There was a lengthy debate on LinkedIn in October 2013, around (the Excel evaluations of) `4^3^2` and `-3^2`.
Note also the MatLab feature that `3i^2`$\Rightarrow -9$ but `3*i^2`$\Rightarrow -3$
The author has seen $1\frac{7}{8}$ mis-OCRed as $\frac{17}{8}$.

Well-enshrined Problems

$0 \in \mathbf{N}$? In theory solved by [ISO], but in practice inconsistent

O When we write $\sin(x) = O(1)$ we really mean $\sin(x) \in O(1)$

Good A few texts are starting to write $\in$

$\sin^2$ When we write $\sin^2 x$ we really mean $(\sin x)^2$

and $\sin^{-1} x$ does not mean $(\sin x)^{-1}$, and $\sin^{-2} x$ is conflicted

Sadly We have to write $\log\log\log x$ because $\log^3 x$ is taken

M_{12} (Entry 1,2 of a matrix, or the 12th Mathieu group?) and the whole `⁣` mess

Lesser problems: metavariables

[AS64, (16.25.1)] defines

$$\mathrm{Pq}(u) = \int_0^u \mathrm{pq}^2(t)\mathrm{d}t$$

but $p, q \in \{s, c, n, d\}$ (so $\mathrm{Sn}(u) = \int_0^u \mathrm{sn}^2(t)\mathrm{d}t$, etc.)

except when q is s, when

$$\mathrm{Pq}(u) = \int_0^u \left(\mathrm{pq}^2(t) - \frac{1}{t^2}\right)\mathrm{d}t - \frac{1}{u},$$

Also "So $\mathcal{X} = (X, \sqsubseteq_X)$ for X equal to T, S, V", defining $\mathcal{T}, \mathcal{S}, \mathcal{V}$ in one go

Lesser problems: surprise

$i = \overline{1:10}$ is being used to mean `for (i=1;i<=10;i++)`, rather than $i = \{1, 2, \ldots, 10\}$.

D_n sometimes means the dihedral group with n elements, sometimes the group on n points (with $2n$ elements)

$\pm$ Is used in many different ways:

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 \pm z_2)}{\cos z_1 \cos z_2}, \qquad \text{[AS64, (4.3.38)]}$$

is shorthand for two equations,

but [AS64, Equations 4.6.26,27]

$$\begin{aligned} \mathrm{Arcsinh}\, z_1 \pm \mathrm{Arcsinh}\, z_2 &= \mathrm{Arcsinh}\left(z_1\sqrt{1+z_2^2} \pm z_2\sqrt{1+z_1^2}\right) \\ \mathrm{Arccosh}\, z_1 \pm \mathrm{Arccosh}\, z_2 &= \mathrm{Arccosh}\left(z_1 z_2 \pm \sqrt{(z_1^2-1)(z_2^2-1)}\right) \end{aligned}$$

is "every value of the left-hand side is a value of the right-hand side and vice-versa"

Recommendations

in increasing order of boldness(!)

- **ISO** Standards exist: the community should follow them
- **And** de facto standards such as [CCN+85, Nat10]
- **Paper** is no longer a scarce resource: some space-saving techniques (metavariables, $\pm$) are actually counterproductive
- **Typesetting** is easy, so $\sin^2 x$ is not cheaper than $(\sin x)^2$, and is notationally polluting
- **$\in$** should be used if that's what we mean
- **Juxtaposition** should be used more sparingly, and properly annotated in MathML

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