Cylindrical Algebraic Decomposition with Logical Structure

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Overview

History of Quantifier Elimination

- In 1930, Tarski discovered [Tar51] that the (semi-)algebraic theory of $\mathbf{R}^n$ admitted quantifier elimination

  $\exists x_{k+1} \forall x_{k+2} \ldots \Phi(x_1, \ldots, x_n) \equiv \Psi(x_1, \ldots, x_k)$
- "Semi" = "allowing $>$, $\leq$ and $\neq$ as well as $=$"
- Needed as $\exists y : x = y^2 \Leftrightarrow x \geq 0$
- The complexity of this was indescribable
- In the sense of not being elementary recursive!
- In 1973, Collins [Col75] discovered a much better way:
- Complexity (m polynomials, degree d, n variables, coefficient length l)

$$(2d)^{2^{2n+8}} m^{2^{n+6}} l^3 \qquad (1)$$

- Construct a cylindrical algebraic decomposition of $\mathbf{R}^n$, sign invariant for every polynomial
- Then read off the answer

What is a CAD?

A Cylindrical Algebraic Decomposition (CAD) is a mathematical object. Defined by Collins who also gave the first algorithm to compute one. A CAD is:

- a decomposition meaning a partition of $\mathbf{R}^n$ into connected subsets called cells;
- (semi-)algebraic meaning that each cell can be defined by a sequence of polynomial equations and inequalities;
- cylindrical meaning the cells are arranged in a useful manner — their projections are either equal or disjoint.

In addition, there is (usually) a sample point in each cell, and an index locating it in the decomposition

“Read off the answer”

- Each cell is sign invariant, so the the truth of a formula throughout the cell is the truth at the sample point.
- $\forall x F(x) \Leftrightarrow$ “$F(x)$ is true at all sample points”
- $\exists x F(x) \Leftrightarrow$ “$F(x)$ is true at some sample point”
- $\forall x \exists y F(x,y) \Leftrightarrow$ “take a CAD of $\mathbf{R}^2$, cylindrical for y projected onto x-space, then check

$$\forall \text{ sample } x \ \exists \text{ sample } (x,y) : F(x,y) \text{ is true''}: \text{ finite check}$$

NB The order of the quantifiers defines the order of projection

So all we need is a CAD!

The basic idea for CAD [Col75]

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[d] \arrow[r, dotted, no head] & \mathbb{R}^n \\
\mathbb{R}^{n-1} \arrow[d] & \mathbb{R}^{n-1} \arrow[u] \\
\mathbb{R}^{n-2} \arrow[d, dashed] & \mathbb{R}^{n-2} \arrow[u] \\
\mathbb{R}^1 \arrow[r] & \mathbb{R}^1 \arrow[u, dashed]
\end{tikzcd}
$$

Projection

Lifting
(& Isolation)

Root Isolation

So how do we project? (Lifting is in fact relatively straight-forward)

Given polynomials $\mathcal{P}_n = \{p_i\}$ in $x_1, \ldots, x_n$, what should $\mathcal{P}_{n-1}$ be?

- **Naïve** (Doesn't work!) Every $\mathrm{Disc}_{x_n}(p_i)$, every $\mathrm{Res}_{x_n}(p_i, p_j)$
- **i.e.** where the polynomials fold, or cross: misses lots of "special" cases
- **[Col75]** First enlarge $\mathcal{P}_n$ with all its reducta, then naïve plus the coefficients of $\mathcal{P}_n$ (with respect to x_n) the principal subresultant coefficients from the Disc_{x_n} and Res_{x_n} calculations
- **[Hon90]** a tidied version of [Col75].
- **[McC88]** Let $\mathcal{B}_n$ be a squarefree basis for the primitive parts of $\mathcal{P}_n$. Then $\mathcal{P}_{n-1}$ is the contents of $\mathcal{P}_n$, the coefficients of $\mathcal{B}_n$ and every $\mathrm{Disc}_{x_n}(b_i)$, $\mathrm{Res}_{x_n}(b_i, b_j)$ from $\mathcal{B}_n$
- **[Bro01]** Naïve plus leading coefficients (not squarefree!)

Setting

Cylindrical Algebraic Decomposition in $\mathbf{R}[x_1, \ldots, x_n]$, with x_n the first variable to be eliminated.
General method via Projection/Lifting in the style of [Col75, Ẅ76].

Open Problem

Extend part 2 of this to the Regular Chains approach [CMXY09]

- [Col75] A cylindrical decomposition of $\mathbf{R}^n$ *sign-invariant* for each polynomial
- [McC84] A cylindrical decomposition of $\mathbf{R}^{n-1}$ *order-invariant* for each polynomial at this stage, and a cylindrical decomposition of $\mathbf{R}^n$ *sign-invariant* for each polynomial
  - or failure if the polynomials were not well-oriented which occurs with probability 0 in theory, but quite often in practice.
- EC An *equational constraint* is $f(\mathbf{x}) = 0 \land \cdots$

Are these projections correct?

[Col75] Yes, and it's relatively straightforward to prove that, over a cell in $\mathbf{R}^{n-1}$ sign-invariant for $\mathcal{P}_{n-1}$, the polynomials of $\mathcal{P}_n$ do not cross, and define cells sign-invariant for the polynomials of $\mathcal{P}_n$

[McC88] 52 pages (based on [Zar75]) prove the equivalent statement, but for order-invariance, not sign-invariance, provided the polynomials are well-oriented, a test that has to be applied during lifting.

But if they're not known to be well-oriented?

[McC88] suggests adding all partial derivatives

In practice hope for well-oriented, and if it fails use Hong's projection.

[Bro01] Needs well-orientedness and additional checks

Motivations for cylindrical algebraic decomposition

1. Quantifier elimination — the original one
   * May have local or global equational constraints
2. Robot Motion Planning — [SS83]
   * Normally has local and global equational constraints
3. Branch Cut analysis [BBDP07]
   * Normally has local equational constraints

Note that we can sometimes transform local ECs into global:

$$(f_1 = 0 \land \phi_1) \lor (f_2 = 0 \land \phi_2)$$

is equivalent to

$$f_1 f_2 = 0 \land [(f_1 = 0 \land \phi_1) \lor (f_2 = 0 \land \phi_2)]$$

Mostly applicable to Quantifier Elimination

Complexity Analysis for [McC84]

Assume m polynomials of degree (in each variable) $\leq d$.
Measure the *number of cells* in the output.
Upper bounds

[McC85, Theorem 6.1.5] $m^{2^n}(2d)^{n2^n}$

[BDE+14, (12)] $2^{2^{n-1}}m(m+1)^{2^n-2}d^{2^n-1}$

* (Same algorithm, better analysis)

Lower bounds (actually of cells in $\mathbf{R}^1$)

[DH88]; $d=4$ $2^{2^{(n-1)/5}}$, and these are the roots of a polynomial of this degree

[BD07]; $d=1$ $2^{2^{(n-1)/3}}$, and in $\mathbf{R}^1$ these are rationals with a succint description.

The original EC observation [Col98, McC99b]

If we have a global equational constraint $f = 0 \wedge \phi$, then all we need is a decomposition that is

1. Sign (or order) invariant for f
2. Sign (or order) invariant for the polynomials g_i of ϕ *when* $f = 0$

Intuitively, we can do this by considering f and $\mathrm{Res}_{x_n}(f, g_i)$ rather than f and g_i for the first projection level, build the order-invariant decomposition of $\mathbf{R}^{n-1}$ for these polynomials (as before), then lift to a sign-invariant decomposition of $\mathbf{R}^n$

Number of cells bounded by [BDE⁺14, (14)]

$$2^{2^{n-1}} d^{2^n-1} m(3m+1)^{2^{n-1}-1},$$

which is "intuitively reasonable" — we can do nothing about degree growth, but combinatorial growth is as for one fewer variable

The theorem that justifies this [McC99b]

Theorem (McCallum1999)

Let f and g be integral polynomials with mvar x_n, and $r(x_1,\ldots,x_{n-1})\neq 0$ be their resultant. Let S be a connected subset of $\mathbf{R}^{n-1}$ on which f is delineable and r order-invariant. Then g is sign-invariant in every section of f over S.

So we can use the McCallum projection

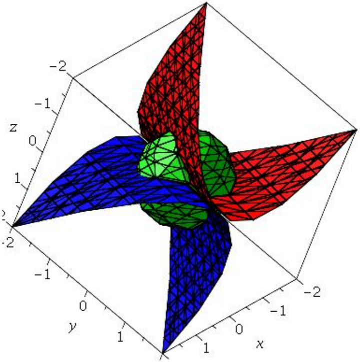
$$P(B) := \mathrm{coeff}(B)\cup \mathrm{Disc}(B)\cup \mathrm{Res}(B)$$

after x_n, where B is the square-free basis of the polynomials, and

$$P_F(B) := P(F)\cup\{\mathrm{Res}(f,g) | f\in F; g\in B\setminus F\}$$

at x_n, where F is the square-free basis of the equational constraint. Note that this theorem does not compose nicely with itself.

Example



$f_1 = x + y^2 + z$
$f_2 = x - y^2 + z$
$g = x^2 + y^2 + z^2 - 1$
$f_1 = 0 \wedge f_2 = 0 \wedge g \geq 0$

Solutions: $y = 0$, $|x| \geq \frac{1}{2}\sqrt{2}$, $z = -x$ (4 cells)
Sign-invariant c.a.d. for $\{f_1, f_2., g\}$ has 1487 cells
Declaring either equational constraint gives 289 cells, but the solution is 8 cells since we have $x = \frac{1}{2}(1 \pm \sqrt{6})$ as additional points from $\mathrm{Disc}_y(\mathrm{Res}_z(f_1, g))$

Part 1: local equational constraints [BDE+13]

Suppose we are doing quantifier elimination on $\phi_1 \vee \phi_2 \vee \cdots$, where each ϕ_i is $f_i = 0 \wedge g_i > 0$ (for simplicity).
There is an implicit equation constraint $F := \prod f_i = 0$, and using [McC99a] our first projection is (ignoring coefficients) $\mathrm{Disc}(F) \cup \{\mathrm{Res}(F, g_i)\}$, which is

$$\{\mathrm{Disc}(f_i)\} \cup \{\mathrm{Res}(f_i, f_j)\} \cup \{\mathrm{Res}(f_i, g_{\color{red}j})\}$$

But this includes $\mathrm{Res}(f_i, g_j)$ $(i \neq j)$, which is logically unnecessary, but is needed to give us a decomposition sign-invariant for each f_i, g_j when $F = 0$.
Relax to demanding a decomposition that's truth-invariant for each ϕ_i:

$$\{\mathrm{Disc}(f_i)\} \cup \{\mathrm{Res}(f_i, f_j)\} \cup \{\mathrm{Res}(f_i, g_{\color{red}i})\}$$

Very useful for the branch cut problem

Local equational constraints continued [BDE+14]

But suppose only *some* ϕ_i have equational constraints, so there isn't a global implicit equational constraint.
Then for those ϕ_i that *do* have an equational constraint $f_i = 0$, the corresponding g_i (possibly many) need only feature in $\mathrm{Res}(f_i, g_i)$: for those ϕ_i with no equational constraint, the g_i feature as usual.

Part 2: A better theorem [McC01]

Theorem (McCallum2001)

Let f and g be integral polynomials with mvar x_n, and $r(x_1,\ldots,x_{n-1})\neq 0$ be their resultant, $d(x_1,\ldots,x_{n-1})\neq 0$ be the discriminant of g. Let S be a connected subset of $\mathbf{R}^{n-1}$ on which f is analytic delineable, g not nullified and r,d order-invariant. Then g is order-invariant in every section of f over S.

This justifies using

$$P_F^*(B) := P_F(B)\cup \mathrm{Disc}(B\setminus F)$$

at levels below x_n where there is an equational constraint, however we need to assume the constraints are primitive.
If we have $f_1=f_2=0$ at x_n, we use $f_1=0$ here, and $\mathrm{Res}(f_1,f_2)$ at level x_{n-1}, etc.
The double exponent of m is reduced by the number of equational constraints.

Better Projection, yes but . . .

Everyone knows that the main cost of c.a.d. is in the lifting. We can also get better lifting, providing we abandon two key principles:

1. That the projection polynomials are a fixed set.
2. That the invariance structure of the final CAD can be expressed in terms of sign-invariance of polynomials.

Idea 1: forget polynomials

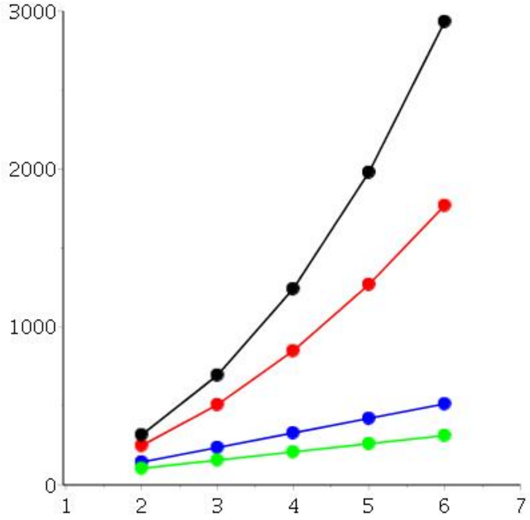
The 1999 theorem states "g is sign-invariant in every section of f over S."

Hence g is unnecessary at the final lift.

Follows from [McC99a], but only noticed in [BDE$^+$13]

Pragmatically very important, but we don't have a theoretical analysis

Idea 1 — Graph of #cells ($n = 2$; $d = 2$; $m = 2 \times x$-axis)



Full CAD
QEPCAD with EC
Our EC with Idea 1
TTICAD

Idea 2: forget sign-invariance

If a cell in $\mathbf{R}^k$ is already known to be false, there is no point doing any (non-trivial) lifting over it.
If we have $f_1 = 0 \wedge f_2 = 0 \wedge \ldots$, then in R^{n-2} we will be looking at the zeros of $\mathrm{Res}_{x_n}(f_1, f_2)$. Away from the zeros of this, $f_1 = 0 \wedge f_2 = 0$ is trivially false, so we needn't do any lifting.
Also, no lifting over C means no nullification worries over C, since this is a *local* concern.

Open Problem

Extend the Phase 2 ideas to merge with Phase 1 (done for some of the lifting reduction)

This seems needed for

Open Problem

Handle non-primitive equational constraints:
$f = 0 \Leftrightarrow \mathrm{pp}_{x_n}(f) = 0 \vee \mathrm{cont}_{x_n}(f) = 0$

Open Problem

Combine this with [BM09] on iterated resultants.

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