The Computer Algebra view of "solving"

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Concepts

Given a set of polynomial equations in $k[x_1, \ldots, x_n]$, how do we solve them

* Or possibly "describe the solutions" if infinitely many

1. Gröbner bases — solves over $\overline{k}$
2. Regular Chains — solves over $\overline{k}$
3. Cylindrical Algebraic Decomposition — solves over $\mathbf{R}$

SMT Of course, we miight only want one solution, or the existence of a solution

Base case

Given A set of linear equations

Reduce to triangular form

$$\begin{pmatrix} 1 & ? & ? & \dots & ? \\ 0 & 1 & ? & \dots & ? \\ 0 & 0 & 1 & \dots & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Solve by back substitution: x_n is obvious, then x_{n-1} is obvious, and so on

Implicitly We've imposed an order on the variables

Nonlinear equations: Order

Ordering the variables is not enough: does x_1^2 come before x_1x_2? Before $x_1x_2^2$? etc.

Lexicographic Sort on x_1 powers, then on x_2 powers ...

Degree lex Sort on total degrees first, then break ties by lex

$$x^3 > x^2y > x^2z > xy^2 > xyz > xz^2 > y^3 > y^2z > yz^2 > z^3$$

Degrevlex Sort on total degrees, then break ties by reverse lex

$$x^3 > x^2y > xy^2 > y^3 > x^2z > xyz > y^2z > xz^2 > yz^2 > z^3$$

Elimination Something on $x_1, \ldots, x_k$, breaking ties on $x_{k+1}, \ldots, x_n$

In general, lexicographic is the most useful, but degrevlex the fastest to compute (but see [vH15])

Beyond Gaussian Elimination

Gaussian can still be done

Also Reduction (division): $x_1x_2 + x_2$ reduces $x_1^2x_2 + x_3$ to $-x_1x_2 + x_3$, which reduces to $x_2 + x_3$ (and this reduces $x_1x_2 + x_2$ to $-x_1x_3 - x_3$)

Insufficient What about $f := x_1^2x_2 + x_2x_3^2$ and $g := x_1x_2^2 + x_4$?

$S(f,g)$ $:= x_2f - x_1g = \cancel{x_1^2}\,\cancel{x_2^2} + x_2^2x_3^2 - (\cancel{x_1^2}\,\cancel{x_2^2} + x_1x_4) = x_2^2x_3^2 - x_1x_4 =: h$

$S(g,h)$ $:= x_3^2g - x_1h = \cancel{x_1}\,\cancel{x_2^2}\,\cancel{x_3^2} + x_3^2x_4 - (\cancel{x_1}\,\cancel{x_2^2}\,/\,x_3^2 - x_1^2x_4) = x_1^2x_4 + x_3x_4$

Note The degree has grown, nevertheless [Buc65] this process terminates, in a *Gröbner basis*

$$[{x_2}^4{x_3}^2 + {x_4}^2, -{x_2}^2{x_3}^2 + x_1\,x_4, x_1\,{x_2}^2 + x_4, {x_1}^2x_2 + x_2\,{x_3}^2]$$

Lex Gröbner bases look like (finitely many solutions)

Generally (Shape Lemma [BMMT94])

$$\begin{pmatrix} x_1 & 0 & 0 & \dots & p_1(x_n) \\ 0 & x_2 & 0 & \dots & p_2(x_n) \\ 0 & 0 & x_3 & \dots & p_3(x_n) \\ \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_n(x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Solve by back-substitution

But not always this shape, e.g.

3 points $\{x_1^2 - 1, x_1(x_2 - 1) - x_2 + 1, x_2^2 - 1\}$

[Gia89, Kal89] Intelligent back-substitution can still work

Also Can convert to Lex by [FGLM93]

Regular Chains/Triangular Decompositions

Regular Chain $T := (f_1, \ldots, f_k)$ such that the f_i have distinct main variables, and each $\mathrm{lt}_{\mathrm{mvar}(f_i)}(f_i)$ is invertible with respect to T .

3 points $\left\{x_1^2 - 1, x_1(x_2 - 1) - x_2 + 1, x_2^2 - 1\right\}$ is not a regular chain

But RCs $\left\{x_1^2 - 1, x_2 - 1\right\}$ and $\{x_1(x_2 - 1) - x_2 + 1, x_2 + 1\}$

Quasivariety $W(T) = V(T) \setminus V\left(\prod \mathrm{lc}_{\mathrm{mvar}(f_i)}(f_i)\right)$: those things that are proved zero by T, without "suspicious cancellation"

(Lazard) **Triangular Decomposition** Produce a set of Regular Chains T_i from F such that $V(F) = \bigcup W(T_i)$

Quantifier Elimination

Throughout, $Q_i \in \{\exists, \forall\}$. Given

$$\Phi := Q_{k+1}x_{k+1} \ldots Q_n x_n \phi(x_1, \ldots, x_n),$$

where ϕ is in some (quantifier-free, generally Boolean-valued) language L, produce an equivalent

$$\Psi := \psi(x_1, \ldots, x_k) : \qquad \psi \in L$$

In particular, $k = 0$ is a decision problem: is Φ true?

Quantifier Elimination is difficult

$$\forall n : n > 1 \Rightarrow \exists p_1 \exists p_2 \left(p_1 \in \mathcal{P} \wedge p_2 \in \mathcal{P} \wedge 2n = p_1 + p_2\right)$$

$$\left[m \in \mathcal{P} \equiv \forall p \forall q \left(m = pq \Rightarrow p = 1 \vee q = 1\right)\right]$$

is a statement of Goldbach's conjecture with, naïvely, seven quantifiers (five will do)

In fact, quantifier elimination is impossible over **N**. [Mat70]

However, it is possible for semi-algebraic (polynomials and inequalities) L over **R** [Tar51]

Unfortunately, the complexity of Tarski's method is indescribable

Over $\mathbf{R}$ we can add $>$ to $=$

(must) $\exists y : y^2 = x \Leftrightarrow x \geq 0$

Hence Semi-algebraic geometry, or real algebraic geometry

CAD "Cylindrical (semi-)Algebraic Decomposition": A partition of $\mathbf{R}^n$ into semi-algebraic sets D_i such that $\forall i, j, k$, if $(x_1, \ldots, x_n) \mapsto_\pi (x_1, \ldots, x_k)$, **either** $\pi(D_i) = \pi(D_j)$ **or** $\pi(D_1) \cap \pi(D_j) = \emptyset$

Also Each D_i has a sample point α_i

Given set f_i of polynomials, construct a CAD *sign-invariant* for every f_i

from a CAD we can read off the answer to any QE problem (quantified in $x_l, \ldots, x_n$ in that order)

Collins' method [Col75]

1. Let $\mathcal{S}_n$ be the polynomials in ϕ (m polynomials, degree d, n variables)
2. Compute $\mathcal{S}_{n-1}$ ($\Theta(m^2)$ polys, degree $\Theta(d^2)$, $n-1$ variables)
3. and $\mathcal{S}_{n-2}$ ($\Theta((m^2)^2)$ polys, degree $\Theta((d^2)^2)$, $n-2$ variables)

$\vdots$ continue

n and $\mathcal{S}_1$ ($\Theta(m^{2^{n-1}})$ polys, degree $\Theta(d^{2^{n-1}})$, 1 variable)

$n+1$ Isolate roots of S_1

$n+2$ Over each root, or interval between roots, isolate roots of S_2

$\vdots$ continue

$2n$ S_n has invariant signs on each region of $\mathbf{R}^n$, so $\phi(x_1,\ldots,x_n)$ has invariant truth on each region

$2n+1$ So evaluate truth of Φ on each region of $(x_1,\ldots,x_k)$-space

Clearly complexity $(md)^{2^{O(n)}}$: in fact $O\left((2m)^{2^{2n+8}}d^{2^{n+6}}\right)$ [Col75]

Collins' method continued

Well, at least that's describable, even if worrying

A better analysis of step $n+1$ [Dav85] gives $O\left((2k)^{2^{2n+\cancel{8}6}} d^{2^{n+\cancel{6}4}}\right)$

which doesn't look very impressive until you realise it's $Z^4 \to Z$

In fact, it largely affects the analysis, not the actual running time

[DH88] showed QE is $\Omega\left(2^{2^{(n-2)/6}}\right)$, or (harder) $\Omega\left(2^{2^{(n-2)/5}}\right)$

(at least in the dense model, i.e. storing all $d+1$ coefficients of a polynomial of degree d).

So we're in $\left(2^{2^{\Theta(n)}}\right)$-land:

this is not the same as $\Theta\left(2^{2^n}\right)$-land, of course

More lower bounds [BD07]

The key idea [Hei83]: suppose Φ_n is $y_n = f_n(x_n)$. Then

$$\Phi_{n+1}(x_{n+1}, y_{n+1}) := \exists z_n \forall x_n \forall y_n$$
$$[(y_n = y_{n+1} \wedge x_n = z_{n+1}) \vee (y_n = z_{n+1} \wedge x_n = x_{n+1})] \Rightarrow \Phi_n(x_n, y_n)$$

is $y_{n+1} = f_n(f_n(x_{n+1}))$. Apply this to

$$f_0(x_0) = \begin{cases} 2x & x \leq 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$$

Then $\Phi_n(x_n, \frac{1}{2})$ defines a set with 2^{2^n} isolated points.
[BD07] shows this set needs doubly exponential space to encode, in dense, sparse or factored form.
However each solution itself is at most singly-exponential ([DH88] has individual solutions doubly-exponential)

Changing the Question

The Heintz construction of [BD07] is $\underbrace{\exists\forall\forall}_{\text{block}} \cdots \underbrace{\exists\forall\forall}_{\text{block}}$, with two **alternations** of quantifiers for every three quantifiers

Let a be the number of alternations

Then [FGM90] the (sequential) cost is $(md)^{n^{O(a)}}$

The doubly-exponential nature is really only for the number of alternations, and it's singly-exponential for the number of variables

I know of no implementation of this method

But It means that cylindrical algebraic decomposition is not always (asymptotically!) best

Order is (sometimes) everything

Consider the polynomial [BD07, Theorem 7]

$$\left((y_{n-1}-\frac{1}{2})^2+(x_{n-1}-z_n)^2\right)\left((y_{n-1}-z_n)^2+(x_{n-1}-x_n)^2\right)x^{n+1}$$
$$+\sum_{i=1}^{n-1}\left((y_{i-1}-y_i)^2+(x_{i-1}-z_i)^2\right)\left((y_{i-1}-z_i)^2+(x_{i-1}-x_i)^2\right)x^{i+1}$$
$$+\left((y_0-2x_0)^2+(\alpha^2+(x_0-\frac{1}{2}))^2\right)\times$$
$$\left((y_0-2+2x_0)^2+(\alpha^2-(x_0-\frac{1}{2}))^2\right)x+a$$

Eliminating $a, x_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}\ldots, z_1, x_0, \alpha, y_0, x$ gives a CAD (in fact a polynomial in a) with at least 2^{2^n} cells, whereas the opposite order has three cells.

Conversely [BD07, Theorem 8] there are problems that are doubly exponential for all orders.

If we can choose the order, how?

Various heuristics:

- **sotd** For all $n!$ orders, perform steps 1-n, measure sotd (sum of total degrees) and do $n+1,\ldots$ for the least
- **Greedy sotd** [DSS04] Do step 1 for each variable, choose the best (sotd) and repeat: often ties
- **ndrr** [BDEW13] For all $n!$ orders, perform steps 1-n, count number of distinct real roots
- **we** tend to use greedy sotd with ndrr as a tiebreaker
- **Brown** [Bro04, 5.2] Eliminate lowest degree variable first (with tie-breaking rules): quite effective

Machine Learning metaheuristic: results from [HEW$^+$14] are encouraging (but what's the benchmark?)

Ordering Example [DSS04]

Lazard's quartic: $\forall x : px^2 + qx + r + x^4 \geq 0$
6 possible orders for (p, q, r)

order	sotd	#cells	CAD	#true	QE
1	54	445	4.71	251	7.04
2	54	445	83.39	251	138.18
3	50	417	0.54	235	0.89
4	50	417	1.64	239	2.55
5	66	—	>600	—	>600
6	66	—	>600	—	>600

Equational Constraints [McC99]

If ϕ is $f = 0 \wedge \hat{\phi}$, we need only consider the cells when $f = 0$ is true. This means the first projection step produces $O(m)$ polynomials rather than $O(m^2)$, and the complexity is
$O\left((2m)^{2^{2n+\cancel{8}6}} d^{2^{n+6}}\right)$.
This gives an interesting formulation problem: given

$$(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0) \qquad (1)$$

we are better off solving the equivalent

$$f_1 f_2 = 0 \wedge [(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0)] \qquad (2)$$

even though the degree goes up: $O\left((2m)^{2^{2n+\cancel{8}6}} d^{2^{n+\cancel{6}7}}\right)$

[There is a technical side-condition *well-orientedness*, possibly obsoleted [MPP16]]

Truth-Table invariant CAD [BDE+16]

In

$$(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0) \tag{3}$$

the first projection set need only be $\mathrm{Disc}(f_1)$, $\mathrm{Disc}(f_2)$, $\mathrm{Res}(f_1, f_2)$, $\mathrm{Res}(f_1, g_1)$, $\mathrm{Res}(f_2, g_2)$ (and omits $\mathrm{Disc}(g_1)$, $\mathrm{Disc}(g_2)$, $\mathrm{Res}(g_1, g_2)$, $\mathrm{Res}(f_1, g_2)$, $\mathrm{Res}(f_1, g_2)$). Essentially all the advantages of equational constraints.

There is still the technical side-condition *well-orientedness*, removed (with many other improvements) in [BCD+14]

There are still issues of formulation: e.g. in $(f_1 = 0 \wedge f_2 = 0 \wedge g_1 < 0) \vee \ldots$, which equation do we prefer?

Alternative method: CAD by Regular Chains [CM14]

C Compute a triangular decomposition over **C**

Hence different challenging problems (may) live in different decompositions

Then Make it *semi-algebraic*, i.e. work out where real lines cross.

Note That this is where different problems interact

Then construct the CAD

Choice of Equational Constraint [BDE+16]

EC Choice 1				EC Choice 2				EC Choice 3			
Cells	Time	S	N	Cells	Time	S	N	Cells	Time	S	N
657	5.6	61	7	463	5.1	64	8	269	1.3	**42**	**4**
711	6.3	66	6	471	5.4	71	6	303	1.1	**40**	**5**
375	2.7	81	9	435	3.6	**73**	**8**	425	2.8	80	**8**
1295	21.4	140	13	477	3.8	**84**	**9**	1437	23.9	158	14
285	2.0	61	7	169	1.0	**59**	**5**				
39	0.1	54	5	9	0.0	**47**	**1**				
F	-	**14**	**0**	F	-	**14**	**0**	27	0.1	**14**	**0**
57	0.3	**32**	**3**	117	0.7	35	**3**	119	0.6	36	4

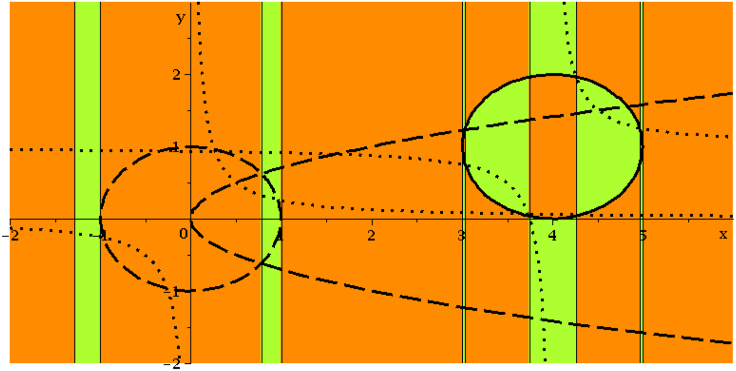
Table: Comparing the choice of equational constraint for a selection of problems. The lowest cell count for each problem is highlighted and the minimal values of the heuristics emboldened.

Which constraint?

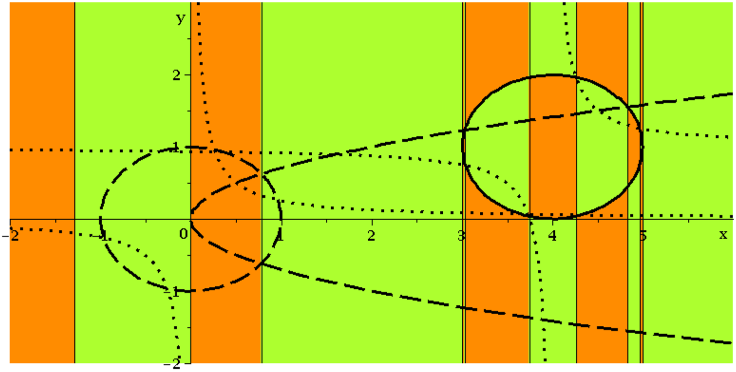
We assume $x \prec y$ and consider $\{\phi_1, \phi_2\}$:

$$\begin{aligned} f_1 &:= x^2 + y^2 - 1, \qquad h := y^2 - \tfrac{x}{2}, \qquad g_1 := xy - \tfrac{1}{4} \\ f_2 &:= (x-4)^2 + (y-1)^2 - 1 \quad g_2 := (x-4)(y-1) - \tfrac{1}{4}, \\ \phi_1 &:= h = 0 \wedge f_1 = 0 \wedge g_1 < 0,\ \phi_2 := f_2 = 0 \wedge g_2 < 0. \end{aligned} \tag{1}$$

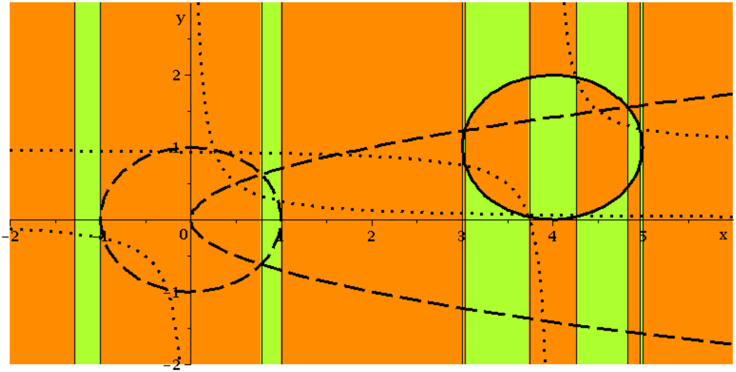
RC-TTICAD with $f_1 \to h \to f_2$ (57 cells).



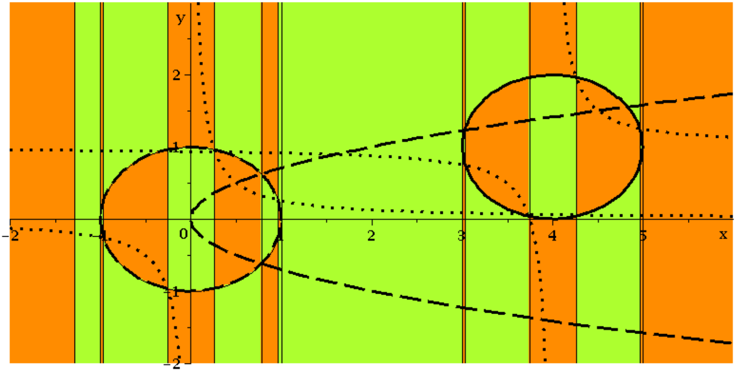
RC-TTICAD with $h \rightarrow f_1 \rightarrow f_2$ (75 cells). This is the default and the same as with f_2, h, f_1.



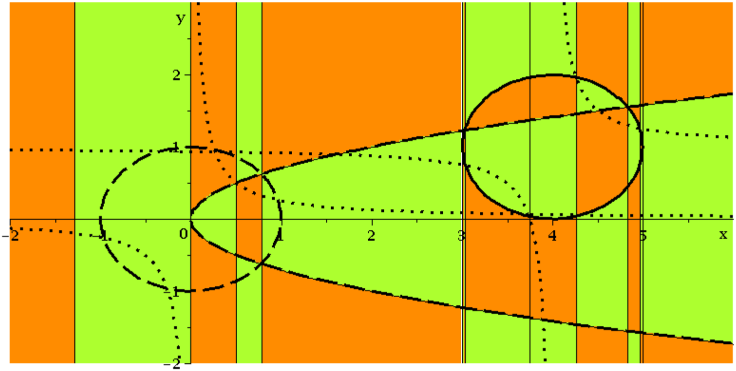
RC-TTICAD with $f_2 \to f_1 \to h$ (77 cells).



PL-TTICAD with f_1 identified (117 cells).



RC-TTICAD with *h* identified (163 cells).



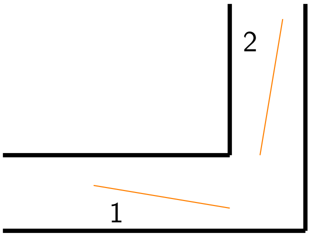
Gröbner Reduction as well [BDEW13]

Order	Full CAD		TTI CAD					TTI+Grö CAD				
	Cells	Time	Eq Const	Cells	Time	S	N	Eq Const	Cells	Time	S	N
$y \prec x$	725	22.802	$f_{1,1}, f_{2,1}$	153	0.818	**62**	12	$\hat{f}_{1,1}, \hat{f}_{2,1}$	27	0.095	**37**	**3**
			$f_{1,1}, f_{2,2}$	111	0.752	94	10	$\hat{f}_{1,1}, \hat{f}_{2,2}$	47	0.361	50	5
			$f_{1,2}, f_{2,1}$	121	0.732	85	9	$\hat{f}_{1,1}, \hat{f}_{2,3}$	93	0.257	50	9
			$f_{1,2}, f_{2,2}$	75	0.840	99	**7**	$\hat{f}_{1,2}, \hat{f}_{2,1}$	47	0.151	47	5
								$\hat{f}_{1,2}, \hat{f}_{2,2}$	83	0.329	63	7
								$\hat{f}_{1,2}, \hat{f}_{2,3}$	145	0.768	81	11
								$\hat{f}_{1,3}, \hat{f}_{2,1}$	95	0.263	46	10
								$\hat{f}_{1,3}, \hat{f}_{2,2}$	151	0.712	80	12
								$\hat{f}_{1,3}, \hat{f}_{2,3}$	209	0.980	62	16
$x \prec y$	657	22.029	$f_{1,1}, f_{2,1}$	125	0.676	**65**	14	$\hat{f}_{1,1}, \hat{f}_{2,1}$	29	0.085	**39**	**4**
			$f_{1,1}, f_{2,2}$	117	0.792	96	11	$\hat{f}_{1,1}, \hat{f}_{2,2}$	53	0.144	52	6
			$f_{1,2}, f_{2,1}$	117	0.728	88	11	$\hat{f}_{1,1}, \hat{f}_{2,3}$	97	0.307	53	97
			$f_{1,2}, f_{2,2}$	85	0.650	101	**8**	$\hat{f}_{1,2}, \hat{f}_{2,1}$	53	0.146	49	6
								$\hat{f}_{1,2}, \hat{f}_{2,2}$	93	0.332	65	8
								$\hat{f}_{1,2}, \hat{f}_{2,3}$	149	0.782	81	13
								$\hat{f}_{1,3}, \hat{f}_{2,1}$	97	0.248	48	11
								$\hat{f}_{1,3}, \hat{f}_{2,2}$	149	0.798	82	13
								$\hat{f}_{1,3}, \hat{f}_{2,3}$	165	1.061	65	18

Table 2. Experimental results relating to Example 7. The lowest cell counts are high

Robot Motion Planning

Reduces to CAD [SS83]. But can we move ladder 1 to position 2?



Insoluble in 1986 [Dav86], insoluble today by [SS83, and today's hardware and CAD advances]

A different formulation [WDEB13]

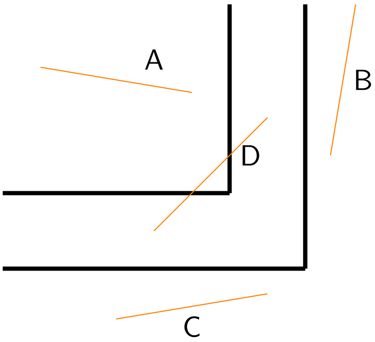


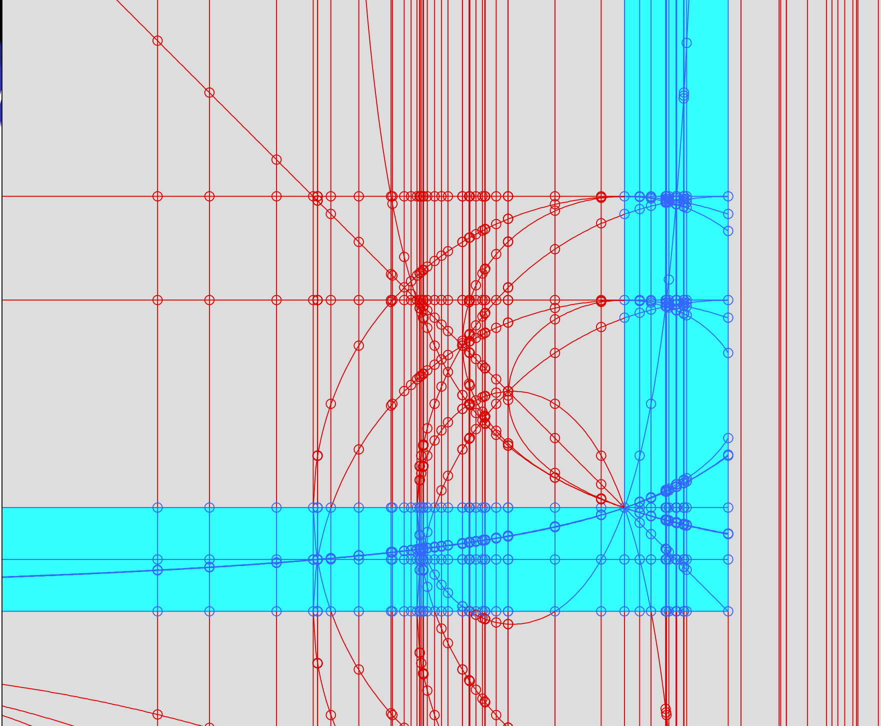
Figure: Four canonical invalid positions of the ladder. Note from the algebraic descriptions that for positions A–C only one end need be outside the corridor.

length$\wedge\neg(A \vee B \vee C \vee D)$: Soluble (5 hours CPU, 285419 cells)

The solution: (but what does it mean?)

$$
\begin{aligned}
& x \leq 0 \wedge y \geq 0 \wedge w \leq 0 \wedge z \geq 0 \wedge (y-z)^2 + (x-w)^2 = 9 \\
& \wedge \Big[[x+1 \geq 0 \wedge w+1 \geq 0] \vee \big[y-1 \leq 0 \wedge w+1 \geq 0 \\
& \qquad \wedge\, y^2w^2 - 2yw^2 + x^2w^2 + 2xw^2 + 2w^2 - 2xy^2w \\
& \qquad\quad + 4xyw - 2x^3w - 4x^2w - 4xw + x^2y^2 - 2x^2y \\
& \qquad\quad + x^4 + 2x^3 - 7x^2 - 18x - 9 \geq 0 \big] \\
& \quad \vee \big[x+1 \geq 0 \wedge yw - w + y + x \geq 0 \wedge w^2 - 2xw + y^2 \\
& \qquad - 2y + x^2 - 8 > 0 \wedge z - 1 \leq 0 \big] \\
& \quad \vee \big[x+1 \geq 0 \wedge yw - w + y + x \geq 0 \wedge y^2w^2 - 2yw^2 \\
& \qquad + x^2w^2 + 2xw^2 + 2w^2 - 2xy^2w + 4xyw - 2x^3w \\
& \qquad\quad - 4x^2w - 4xw + x^2y^2 - 2x^2y + x^4 + 2x^3 - 7x^2 \\
& \qquad\quad - 18x - 9 \leq 0 \wedge z - 1 \leq 0 \big] \\
& \quad \vee [y - 1 \leq 0 \wedge z - 1 \leq 0] \Big].
\end{aligned} \tag{4}
$$

How



Conclusions

The more I learn, the less I know, but

- There's more than one way to state a problem
- Clearly equivalent in terms of **decidability**, but not **practical computability**
- The differences are vast in practice
- We have some reasonable heuristics
- But much more work needs to be done, theoretically, experimentally, and on the "software packaging" side
- We need practical work on alternative methods for quantifier elimination

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