Symbolic Computation (Computer Algebra)

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Generally talking about “polynomial” computer algebra:
Major packages: Maple and Mathematica (commercial); Reduce, Macsyma; meta-package Sage
There’s also the group theory end of the world: GAP and MAGMA
Many specialist packages: F5, Singular and CoCoA for Gröbner bases, QEPCAD, RedLog for real solving, and I’ve doubtless omitted many others
Notation (for complexity purposes)

- $m$ number of polynomials
- $n$ number of variables
- $d$ maximum degree (in each variable separately)
- $l$ bit-length of coefficients (we will often not bother with this, as most solving algorithms are $O(l^3)$)
- $t$ Number of non-zero terms.
“Obviously” a vector of coefficients

**Addition** $O(dl)$

**Multiplication** School: $O(d^2l^2)$; Karatsuba: $O((dl)^{1.585})$; FFT: $O(dl \log(dl) \log \log l)$

**Division** the same

**GCD** the same $\times \log d$

But most problems aren’t dense: a dense polynomial has $(d + 1)^n$ terms.
“Obviously” a list of (exponent,coefficient) pairs

**Addition** $O(tl)$

**Multiplication School:** $O(t^3 + t^2l^2)$; Better $O(t^2(l^2 + \log t))$

**Division** not the same at all: $\frac{x^{n-1}}{x-1} = x^{n-1} + \cdots + 1$

**D with remainder School:** $O(d^3tl^2)$, better $O(d^2t(l^2 + \log d))$

**Exact D** We can stop if the coefficients grow too big [ABD85] $O(dt(dl^2 + \log \min(d, t)))$; in fact $O(t_f + t_{f/g}t_g(l^2 + \log \min(t_{f/g}t_g)))$ [MP11]

**GCD** $O(d^4l^2)$: some dependence on $d$ is inherent [Sch03]:

$$\gcd(x^{pq}-1, x^{p+q}-x^p-x^q+1) = x^{p+q-1} - x^{p+q-2} \pm \cdots - 1$$

**Open** An algorithm for $\gcd(f,g)$ polynomial in $t_f, t_g, t_{\gcd(f,g)}$. 
In practice

We compute polynomial gcd via modular methods

1. Compute gcd($f$, $g$) modulo several $p_i$
   
   - How many? *A priori* bounds are usually too high, so the best algorithms are adaptive

2. Discard those that have too high a degree

3. Use Chinese Remainder Theorem to produce gcd($f$, $g$) (mod $\prod' p_i$)

4. Interpret over $\mathbb{Z}$ and check

Same technique works in several variables, using $y - v_i$ as “primes”

The real challenge is multivariate sparsity: [Zip79] has an algorithms that seems to be polynomial in $d, t$ (and not $d^n$)
Can’t use modular methods, as no idea which factor goes with which \((\mathbb{Z}[x]/\prod p_i)\) is not a unique factorisation domain

There are irreducible polynomials (e.g. \(x^4 + 1\)) which factor compatibly modulo every prime (no good primes!)

rare in theory but common in practice [ABD85], especially with algebraic numbers

Trying every combination of mod \(p\) factors is exponential

LLL was invented to make this polynomial, but \(O(d^{12})\)

There are better lattice-based methods these days [vH02]
Polynomial Factoring (multivariate)

- For reductions from multivariate to bivariate
  (by replacing all the other variables by values)
  almost all evaluations are good, in that the factorisation is the same after evaluating
- In practice we see the same reducing to univariate

Hence the real challenge is univariate, in theory

In practice there are substantial challenges especially over sparsity [Wan78]

But Factoring is still best avoided, and implementations try to:
  replacing “irreducible” by “square-free and relatively prime”
  where possible
Univariate  A polynomial has $d$ roots

Computed  Sturm sequences, Thom’s lemma, Descartes Rule

Sparse    A polynomial has $\leq 2t - 1$ real roots

(Almost) no good, i.e. $O(t^k)$ not $O(d^k)$, algorithms for isolating these

Multivariate again, should have low real complexity, but “change of coordinates” destroys sparsity.
Resultants and Discriminants

Resultant \( R(x) := \text{Res}_y(f(x, y), g(x, y)) \) has as roots those \( x : \exists y \in \mathbb{C} : f(x, y) = g(x, y) = 0 \): common zeros = crossings of surfaces

Discriminant \( D(x) = \text{Disc}_y(f(x, y)) = \frac{1}{\text{lc}_y(f)} \text{Res}(f, \frac{\partial f}{\partial y}) \) has as roots those \( x : \exists y \in \mathbb{C} : f(x, y) \) has a double root: self-crossings or doubling-back

Degrees \( O(d^2) \) and classical computing time \( O(d^9 l^2) \) — modular methods are generally used, but still expensive

\( \text{Res}(f_1 f_2, g) = \text{Res}(f_1, g) \text{Res}(f_2, g); \)
\( \text{Disc}(f_1 f_2) = \text{Disc}(f_1) \text{Disc}(f_2) \text{Res}^2(f_1, f_2), \) hence we want to keep polynomials in factored form

Iterated resultants also tend to factor also [BM09], generally into a “good part” and a “bad part”

No good interaction with sparsity!
Gröbner Bases

Many views and uses, but we can consider them as non-linear analogue of Gaussian reduction to upper triangular form, **but**:

- Might have more equations than variables:
  \[\{x^2 - 1, (x - 1)(y - 1), y^2 - 1\} \text{ — Back-substitution by Gianni–Kalkbrener Theorem [Gia89, Kal89]}\]
  
- This doesn’t work in dimension > 0 [FGT01]

- Can have “mixed dimension” varieties:
  \[\langle (x + 1 - y)(x - 6 + y), (x + 1 - y)(y - 3) \rangle\] has solution \[x = y - 1 \text{ and } (x, y) = (3, 3)\]

- Theoretical complexity results are very bad, but these inputs are “rare”

- Modular (Chinese Remainder) approaches are difficult
In 1930, Tarski discovered [Tar51] that the (semi-)algebraic theory of $\mathbb{R}^n$ admitted quantifier elimination

$$\exists x_{k+1} \forall x_{k+2} \ldots \Phi(x_1, \ldots, x_n) \equiv \Psi(x_1, \ldots, x_k)$$

“Semi” = “allowing $>$, $\leq$ and $\neq$ as well as $=$”

Needed as $\exists y : x = y^2 \iff x \geq 0$

The complexity of this was indescribable

In the sense of not being elementary recursive!

In 1973, Collins [Col75] discovered a much better way:

Complexity ($m$ polynomials, degree $d$, $n$ variables, coefficient length $l$)

$$\text{(2d)}^{2n^8} m^{2n^6} l^3$$

Construct a cylindrical algebraic decomposition of $\mathbb{R}^n$, sign invariant for every polynomial

Then read off the answer
What is a CAD?

A Cylindrical Algebraic Decomposition (CAD) is a mathematical object. Defined by Collins who also gave the first algorithm to compute one. A CAD is:

- a decomposition meaning a partition of $\mathbb{R}^n$ into connected subsets called cells;
- (semi-)algebraic meaning that each cell can be defined by a sequence of polynomial equations and inequalities;
- cylindrical meaning the cells are arranged in a useful manner — their projections are either equal or disjoint.

In addition, there is (usually) a sample point in each cell, and an index locating it in the decomposition.
Each cell is sign invariant, so the truth of a formula throughout the cell is the truth at the sample point.

- \( \forall x F(x) \iff \text{"}F(x)\text{" is true at all sample points}" \)
- \( \exists x F(x) \iff \text{"}F(x)\text{" is true at some sample point}" \)
- \( \forall x \exists y F(x, y) \iff \text{“take a CAD of } \mathbb{R}^2, \text{ cylindrical for } y \text{ projected onto } x\text{-space, then check} \)\
  \[ \forall \text{ sample } x \exists \text{ sample } (x, y) : F(x, y) \text{ is true”}: \text{ finite check} \]

NB The order of the quantifiers defines the order of projection

So all we need is a CAD!
The basic idea for CAD [Col75]

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So how do we project?
(Lifting is in fact relatively straight-forward)

Given polynomials \( P_n = \{p_i\} \) in \( x_1, \ldots, x_n \), what should \( P_{n-1} \) be?

Naïve (Doesn’t work!) Every \( \text{Disc}_{x_n}(p_i) \), every \( \text{Res}_{x_n}(p_i, p_j) \)

i.e. where the polynomials fold, or cross: misses lots of “special” cases

[Col75] First enlarge \( P_n \) with all its reducta, then naïve plus
the coefficients of \( P_n \) (with respect to \( x_n \)) the
principal subresultant coefficients from the \( \text{Disc}_{x_n} \)
and \( \text{Res}_{x_n} \) calculations

[Hon90] a tidied version of [Col75].

[McC88] Let \( B_n \) be a squarefree basis for the primitive parts of
\( P_n \). Then \( P_{n-1} \) is the contents of \( P_n \), the coefficients
of \( B_n \) and every \( \text{Disc}_{x_n}(b_i), \text{Res}_{x_n}(b_i, b_j) \) from \( B_n \)

[Bro01] Naïve plus leading coefficients (not squarefree!)
Are these projections correct?

Yes, and it’s relatively straightforward to prove that, over a cell in $\mathbb{R}^{n-1}$ sign-invariant for $P_{n-1}$, the polynomials of $P_n$ do not cross, and define cells sign-invariant for the polynomials of $P_n$. 

52 pages (based on [Zar75]) prove the equivalent statement, but for order-invariance, not sign-invariance, provided the polynomials are well-oriented, a test that has to be applied during lifting.

But if they’re not known to be well-oriented?

suggests adding all partial derivatives

In practice hope for well-oriented, and if it fails use Hong’s projection.

Needs well-orientedness and additional checks
What about the complexity?

If the McCallum projection is well-oriented, the complexity is

\[(2d)^{n^2n^+7} m^{2n^+4} l^3\]  \hspace{1cm} (2)

versus the original

\[(2d)^{2n^+8} m^{2n^+6} l^3\]  \hspace{1cm} (1)

and in practice the gains in running time can be factors of a thousand, or, more often, the difference between feasibility and infeasibility.

“Randomly”, well-orientedness ought to occur with probability 1, but we have a family of “real-world” examples (simplification/branch cuts) where it often fails.
The Heintz construction

\[ \Phi_k(x_k, y_k) := \exists z_k \forall x_{k-1} y_{k-1} \left[ y_{k-1} = y_k \land x_{k-1} = z_k \lor y_{k-1} = z_k \land x_{k-1} = x_k \Rightarrow \Phi_{k-1}(x_{k-1}, y_{k-1}) \right] \]

If \( \Phi_1 \equiv y_1 = f(x_1) \), then \( \Phi_2 \equiv y_2 = f(f(x_2)) \), 
\( \Phi_3 \equiv y_3 = f(f(f(f(x_3)))) \)

[DH88] shows \( \Omega \left( 2^{2^{(n-2)/5}} \right) \) (using \( y_R + iy_I = (x_R + ix_I)^4 \))

[BD07] shows \( \Omega \left( 2^{2^{(n-1)/3}} \right) \) (using a sawtooth)

Hence doubly exponential is inevitable, but there’s a lot of room! 
In fact, there are theoretical algorithms which are singly-exponential in \( n \), but doubly-exponential in the number of \( \exists \forall \) alternations
Useful special cases

[McC99] “equational constraints” : when
\[ \Phi \equiv f(x, y, \ldots) = 0 \land (\ldots) \]

Note If \[ \Phi \equiv (f_1(x, y) = 0 \land g_1(x, y) < 0) \lor (f_2(x, y) = 0 \land g_2(x, y) < 0), \]
which has no obvious equational constraint, we can consider
\[ (f_1 \cdot f_2)(x, y) = 0 \land \Phi, \]
which is equivalent (but higher degree)

[BDE+13] “truth table invariant CAD” treats this directly

[BDE+14] also handles the case where not every clause has an equality (TTICAD)

Roughly speaking, the effect is to reduce \( n \) by 1, \textit{which square roots the complexity}
An alternative approach [CMXY09]

Proceed via the complex numbers,

\[ \mathbb{C}^n \rightarrow \mathbb{C}^n \]

\[ \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \]

\[ \mathbb{R}^1 \rightarrow \mathbb{R}^1 \]

Do a complex cylindrical decomposition via Regular Chains
Can be combined with truth table ideas [EBC^{+}14]
Figure: Complete complex cylindrical tree for the general monic quadratic equation, $p := x^2 + bx + c$, under variable ordering $c \prec b \prec x$.

Note that $b = 0$ is only tested where relevant.
So how do I use these tools?

That’s actually a very good question: there’s a lot of choice in how to phrase the question

1. Choice of variable ordering (where permitted)
2. Choice of equalities
3. Choice of overall technology (Projection/Regular Chains/…)
4. Choice of how the problem is posed
5. (including Gröbner pre-conditioning)

Choice of software: no software has (even close to) all the techniques, and each has extra “features”

These are not independent questions
How might this look? Wilson’s thesis

Description of Problem

Projection & Lifting
  - EC/TTICAD

Variable Ordering
  - Preconditioning
  - Designation
  - Layered

Regular Chains
  - EC/TTICAD

Variable Ordering
  - Preconditioning
  - Designation
  - Layered
Variable ordering

Theorem ([BD07])

*There are CAD problems doubly exponential (in n) for all orderings, and other problems which are doubly exponential (in n) for some orderings, but constant for others*

How to tell which case we’re in?
How to choose the best (legal) ordering?
This was described in [HEW⁺14]:
a variety of heuristics, with a machine-learning meta-heuristic


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