Formulating Problems in Real Algebra/Geometry

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Quantifier Elimination

Throughout, $Q_i \in \{\exists, \forall\}$. Given

$$\Phi := Q_{k+1} x_{k+1} \dots Q_n x_n \phi(x_1, \dots, x_n),$$

where ϕ is in some (quantifier-free, generally Boolean-valued) language L, produce an equivalent

$$\Psi := \psi(x_1, \dots, x_k) : \qquad \psi \in L$$

In particular, $k = 0$ is a decision problem: is Φ true?

Quantifier Elimination is difficult

$$\forall n : n > 1 \Rightarrow \exists p_1 \exists p_2 \left(p_1 \in \mathcal{P} \wedge p_2 \in \mathcal{P} \wedge 2n = p_1 + p_2 \right)$$

$$\left[m \in \mathcal{P} \equiv \forall p \forall q \left(m = pq \Rightarrow p = 1 \vee q = 1 \right) \right]$$

is a statement of Goldbach's conjecture with, naïvely, seven quantifiers (five will do)

In fact, quantifier elimination is impossible over **N**. [Mat70]

However, it is possible for semi-algebraic (polynomials and inequalities) L over **R** [Tar51]

Unfortunately, the complexity of Tarski's method is indescribable

Collins' method [Col75]

1. Let $\mathcal{S}_n$ be the polynomials in ϕ (m polynomials, degree d, n variables)
2. Compute $\mathcal{S}_{n-1}$ ($\Theta(m^2)$ polys, degree $\Theta(d^2)$, $n-1$ variables)
3. and $\mathcal{S}_{n-2}$ ($\Theta((m^2)^2)$ polys, degree $\Theta((d^2)^2)$, $n-2$ variables)

$\vdots$ continue

n and $\mathcal{S}_1$ ($\Theta(m^{2^{n-1}})$ polys, degree $\Theta(d^{2^{n-1}})$, 1 variable)

$n+1$ Isolate roots of S_1

$n+2$ Over each root, or interval between roots, isolate roots of S_2

$\vdots$ continue

$2n$ S_n has invariant signs on each region of $\mathbf{R}^n$, so $\phi(x_1, \ldots, x_n)$ has invariant truth on each region

$2n+1$ So evaluate truth of Φ on each region of $(x_1, \ldots, x_k)$-space

Clearly complexity $(md)^{2^{O(n)}}$: in fact $O\left((2m)^{2^{2n+8}} d^{2^{n+6}}\right)$ [Col75]

Collins' method continued

Well, at least that's describable, even if worrying

A better analysis of step $n+1$ [Dav85] gives $O\left((2k)^{2^{2n+\cancel{8}6}} d^{2^{n+\cancel{6}4}}\right)$

which doesn't look very impressive until you realise it's $Z^4 \to Z$

In fact, it largely affects the analysis, not the actual running time

[DH88] showed QE is $\Omega\left(2^{2^{(n-2)/6}}\right)$, or (harder) $\Omega\left(2^{2^{(n-2)/5}}\right)$

(at least in the dense model, i.e. storing all $d+1$ coefficients of a polynomial of degree d).

So we're in $\left(2^{2^{\Theta(n)}}\right)$-land:

this is not the same as $\Theta\left(2^{2^n}\right)$-land, of course

More lower bounds [BD07]

The key idea [Hei83]: suppose Φ_n is $y_n = f_n(x_n)$. Then

$$\Phi_{n+1}(x_{n+1}, y_{n+1}) := \exists z_n \forall x_n \forall y_n$$
$$[(y_n = y_{n+1} \wedge x_n = z_{n+1}) \vee (y_n = z_{n+1} \wedge x_n = x_{n+1})] \Rightarrow \Phi_n(x_n, y_n)$$

is $y_{n+1} = f_n(f_n(x_{n+1}))$. Apply this to

$$f_0(x_0) = \begin{cases} 2x & x \leq 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$$

Then $\Phi_n(x_n, \frac{1}{2})$ defines a set with 2^{2^n} isolated points.
[BD07] shows this set needs doubly exponential space to encode, in dense, sparse or factored form.

Changing the Question

The Heintz construction of [BD07] is $\underbrace{\exists\forall\forall}_{\text{block}} \cdots \underbrace{\exists\forall\forall}_{\text{block}}$, with two **alternations** of quantifiers for every three quantifiers

Let a be the number of alternations

Then [FGM90] the (sequential) cost is $(md)^{n^{O(a)}}$

The doubly-exponential nature is really only for the number of alternations, and it's singly-exponential for the number of variables

I know of no implementation of this method

But It means that cylindrical algebraic decomposition is not always (asymptotically!) best

Order is (sometimes) everything

Consider the polynomial [BD07, Theorem 7]

$$\left((y_{n-1}-\frac{1}{2})^2+(x_{n-1}-z_n)^2\right)\left((y_{n-1}-z_n)^2+(x_{n-1}-x_n)^2\right)x^{n+1}$$
$$+\sum_{i=1}^{n-1}\left((y_{i-1}-y_i)^2+(x_{i-1}-z_i)^2\right)\left((y_{i-1}-z_i)^2+(x_{i-1}-x_i)^2\right)x^{i+1}$$
$$+\left((y_0-2x_0)^2+(\alpha^2+(x_0-\frac{1}{2}))^2\right)\times$$
$$\left((y_0-2+2x_0)^2+(\alpha^2-(x_0-\frac{1}{2}))^2\right)x+a$$

Eliminating $a, x_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}\ldots, z_1, x_0, \alpha, y_0, x$ gives a CAD (in fact a polynomial in a) with at least 2^{2^n} cells, whereas the opposite order has three cells.

Conversely [BD07, Theorem 8] there are problems that are doubly exponential for all orders.

If we can choose the order, how?

Various heuristics:

- **sotd** For all $n!$ orders, perform steps 1-n, measure sotd (sum of total degrees) and do $n+1,\ldots$ for the least
- **Greedy sotd** [DSS04] Do step 1 for each variable, choose the best (sotd) and repeat: often ties
- **ndrr** [BDEW13] For all $n!$ orders, perform steps 1-n, count number of distinct real roots
- **we** tend to use greedy sotd with ndrr as a tiebreaker
- **Brown** [Bro04, 5.2] Eliminate lowest degree variable first (with tie-breaking rules): quite effective
- **Machine Learning** metaheuristic: very preliminary results from Zongyan Huang (Cambridge) are encouraging

Ordering Example [DSS04]

Lazard's quartic: $\forall x : px^2 + qx + r + x^4 \geq 0$
6 possible orders for (p, q, r)

order	sotd	#cells	CAD	#true	QE
1	54	445	4.71	251	7.04
2	54	445	83.39	251	138.18
3	50	417	0.54	235	0.89
4	50	417	1.64	239	2.55
5	66	—	$>$600	—	$>$600
6	66	—	$>$600	—	$>$600

Equational Constraints [McC99]

If ϕ is $f = 0 \wedge \hat{\phi}$, we need only consider the cells when $f = 0$ is true. This means the first projection step produces $O(m)$ polynomials rather than $O(m^2)$, and the complexity is $O\left((2m)^{2^{2n+\cancel{8}6}} d^{2^{n+6}}\right)$.

This gives an interesting formulation problem: given

$$(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0) \tag{1}$$

we are better off solving the equivalent

$$f_1 f_2 = 0 \wedge [(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0)] \tag{2}$$

even though the degree goes up: $O\left((2m)^{2^{2n+\cancel{8}6}} d^{2^{n+\cancel{6}7}}\right)$

[There is a technical side-condition *well-orientedness*]

Truth-Table invariant CAD [BDE+13]

In

$$(f_1 = 0 \wedge g_1 < 0) \vee (f_2 = 0 \wedge g_2 < 0) \qquad (3)$$

the first projection set need only be $\mathrm{Disc}(f_1)$, $\mathrm{Disc}(f_2)$, $\mathrm{Res}(f_1, f_2)$, $\mathrm{Res}(f_1, g_1)$, $\mathrm{Res}(f_2, g_2)$ (and omits $\mathrm{Disc}(g_1)$, $\mathrm{Disc}(g_2)$, $\mathrm{Res}(g_1, g_2)$, $\mathrm{Res}(f_1, g_2)$, $\mathrm{Res}(f_1, g_2)$). Essentially all the advantages of equational constraints.

There is still the technical side-condition *well-orientedness*, removed (with many other improvements) in [BCD+14]

There are still issues of formulation: e.g. in $(f_1 = 0 \wedge f_2 = 0 \wedge g_1 < 0) \vee \ldots$, which equation do we prefer?

Choice of Equational Constraint [BDEW13]

EC Choice 1				EC Choice 2				EC Choice 3			
Cells	Time	S	N	Cells	Time	S	N	Cells	Time	S	N
657	5.6	61	7	463	5.1	64	8	269	1.3	**42**	**4**
711	6.3	66	6	471	5.4	71	6	303	1.1	**40**	**5**
375	2.7	81	9	435	3.6	**73**	**8**	425	2.8	80	**8**
1295	21.4	140	13	477	3.8	**84**	**9**	1437	23.9	158	14
285	2.0	61	7	169	1.0	**59**	**5**				
39	0.1	54	5	9	0.0	**47**	**1**				
F	-	**14**	**0**	F	-	**14**	**0**	27	0.1	**14**	**0**
57	0.3	**32**	**3**	117	0.7	35	**3**	119	0.6	36	4

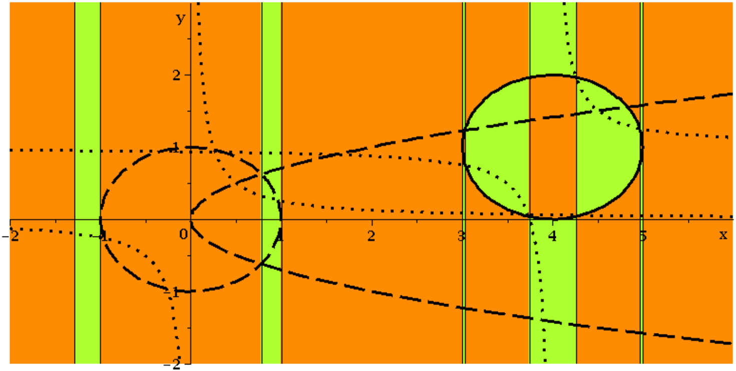
Table: Comparing the choice of equational constraint for a selection of problems. The lowest cell count for each problem is highlighted and the minimal values of the heuristics emboldened.

Which constraint?

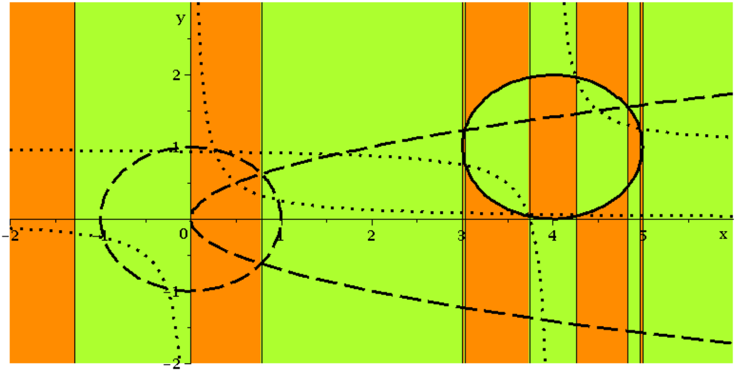
We assume $x \prec y$ and consider $\{\phi_1, \phi_2\}$:

$$\begin{aligned}
&f_1 := x^2 + y^2 - 1, \qquad h := y^2 - \tfrac{x}{2}, \qquad g_1 := xy - \tfrac{1}{4} \\
&f_2 := (x-4)^2 + (y-1)^2 - 1 \quad g_2 := (x-4)(y-1) - \tfrac{1}{4}, \\
&\phi_1 := h = 0 \wedge f_1 = 0 \wedge g_1 < 0, \;\; \phi_2 := f_2 = 0 \wedge g_2 < 0.
\end{aligned} \tag{1}$$

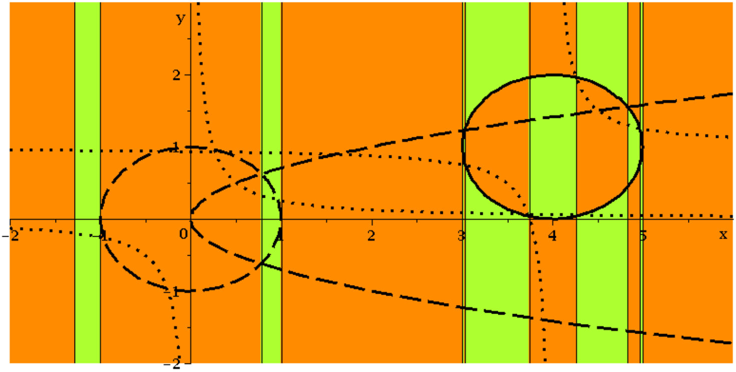
RC-TTICAD with $f_1 \to h \to f_2$ (57 cells).



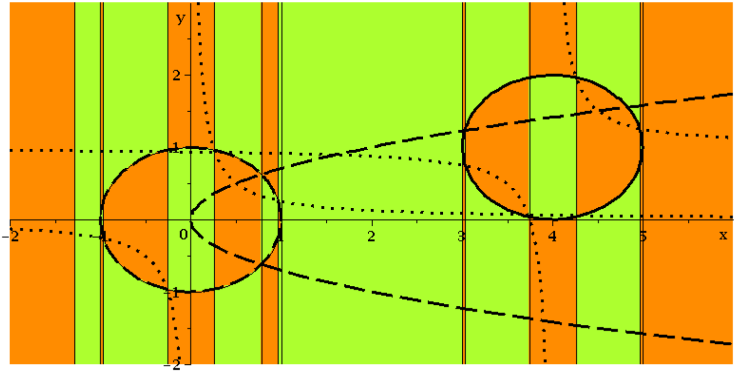
RC-TTICAD with $h \to f_1 \to f_2$ (75 cells). This is the default and the same as with f_2, h, f_1.



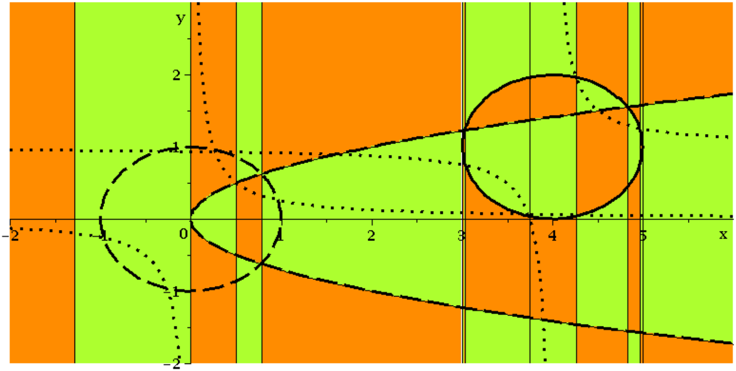
RC-TTICAD with $f_2 \to f_1 \to h$ (77 cells).



PL-TTICAD with f_1 identified (117 cells).



RC-TTICAD with h identified (163 cells).



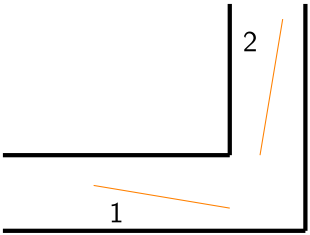
Gröbner Reduction as well [BDEW13]

Order	Full CAD		TTI CAD					TTI+Grö CAD				
	Cells	Time	Eq Const	Cells	Time	S	N	Eq Const	Cells	Time	S	N
$y \prec x$	725	22.802	$f_{1,1}, f_{2,1}$	153	0.818	**62**	12	$\hat{f}_{1,1}, \hat{f}_{2,1}$	27	0.095	**37**	**3**
			$f_{1,1}, f_{2,2}$	111	0.752	94	10	$\hat{f}_{1,1}, \hat{f}_{2,2}$	47	0.361	50	5
			$f_{1,2}, f_{2,1}$	121	0.732	85	9	$\hat{f}_{1,1}, \hat{f}_{2,3}$	93	0.257	50	9
			$f_{1,2}, f_{2,2}$	75	0.840	99	**7**	$\hat{f}_{1,2}, \hat{f}_{2,1}$	47	0.151	47	5
								$\hat{f}_{1,2}, \hat{f}_{2,2}$	83	0.329	63	7
								$\hat{f}_{1,2}, \hat{f}_{2,3}$	145	0.768	81	11
								$\hat{f}_{1,3}, \hat{f}_{2,1}$	95	0.263	46	10
								$\hat{f}_{1,3}, \hat{f}_{2,2}$	151	0.712	80	12
								$\hat{f}_{1,3}, \hat{f}_{2,3}$	209	0.980	62	16
$x \prec y$	657	22.029	$f_{1,1}, f_{2,1}$	125	0.676	**65**	14	$\hat{f}_{1,1}, \hat{f}_{2,1}$	29	0.085	**39**	**4**
			$f_{1,1}, f_{2,2}$	117	0.792	96	11	$\hat{f}_{1,1}, \hat{f}_{2,2}$	53	0.144	52	6
			$f_{1,2}, f_{2,1}$	117	0.728	88	11	$\hat{f}_{1,1}, \hat{f}_{2,3}$	97	0.307	53	97
			$f_{1,2}, f_{2,2}$	85	0.650	101	**8**	$\hat{f}_{1,2}, \hat{f}_{2,1}$	53	0.146	49	6
								$\hat{f}_{1,2}, \hat{f}_{2,2}$	93	0.332	65	8
								$\hat{f}_{1,2}, \hat{f}_{2,3}$	149	0.782	81	13
								$\hat{f}_{1,3}, \hat{f}_{2,1}$	97	0.248	48	11
								$\hat{f}_{1,3}, \hat{f}_{2,2}$	149	0.798	82	13
								$\hat{f}_{1,3}, \hat{f}_{2,3}$	165	1.061	65	18

Table 2 Experimental results relating to Example 7. The lowest cell counts are high

Robot Motion Planning

Reduces to CAD [SS83]. But can we move ladder 1 to position 2?



Insoluble in 1986 [Dav86], insoluble today by [SS83, and today's hardware and CAD advances]

A different formulation [WBDE13]

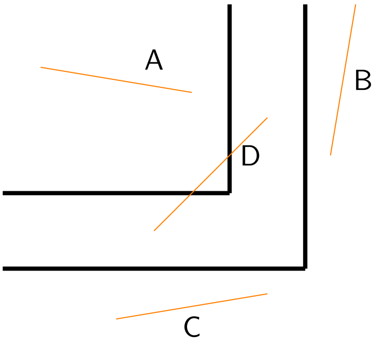


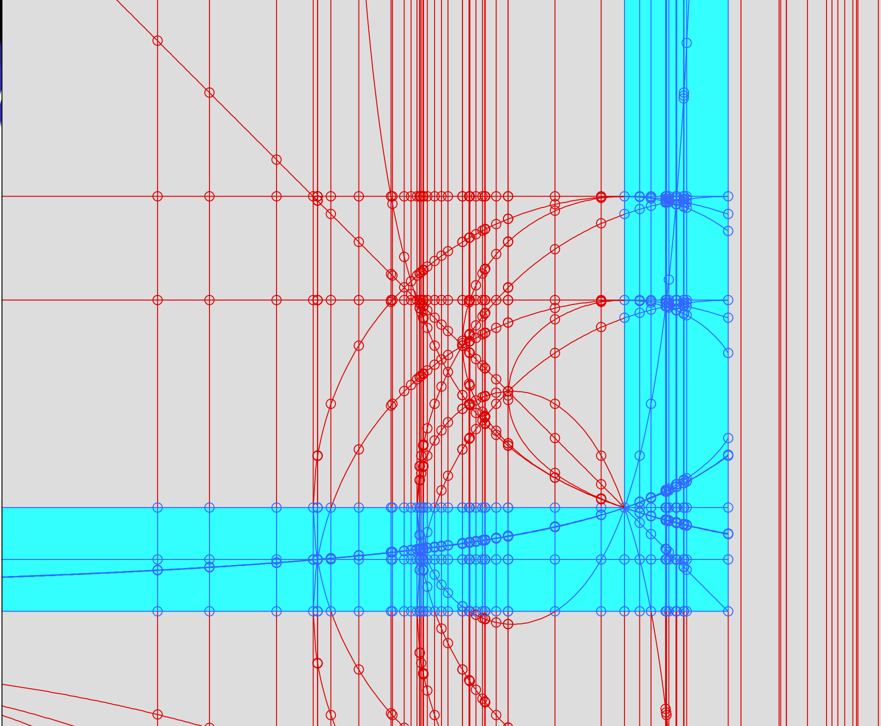
Figure: Four canonical invalid positions of the ladder. Note from the algebraic descriptions that for positions A–C only one end need be outside the corridor.

length$\wedge\neg(A \vee B \vee C \vee D)$: Soluble (5 hours CPU, 285419 cells)

The solution:

$$
\begin{aligned}
&x \leq 0 \wedge y \geq 0 \wedge w \leq 0 \wedge z \geq 0 \wedge (y-z)^2 + (x-w)^2 = 9 \\
&\wedge \Big[[x+1 \geq 0 \wedge w+1 \geq 0] \vee \big[y-1 \leq 0 \wedge w+1 \geq 0 \\
&\qquad \wedge y^2w^2 - 2yw^2 + x^2w^2 + 2xw^2 + 2w^2 - 2xy^2w \\
&\qquad\quad + 4xyw - 2x^3w - 4x^2w - 4xw + x^2y^2 - 2x^2y \\
&\qquad\quad + x^4 + 2x^3 - 7x^2 - 18x - 9 \geq 0 \big] \\
&\quad \vee \big[x+1 \geq 0 \wedge yw - w + y + x \geq 0 \wedge w^2 - 2xw + y^2 \\
&\qquad - 2y + x^2 - 8 > 0 \wedge z - 1 \leq 0 \big] \\
&\quad \vee \big[x+1 \geq 0 \wedge yw - w + y + x \geq 0 \wedge y^2w^2 - 2yw^2 \\
&\qquad + x^2w^2 + 2xw^2 + 2w^2 - 2xy^2w + 4xyw - 2x^3w \\
&\qquad\quad - 4x^2w - 4xw + x^2y^2 - 2x^2y + x^4 + 2x^3 - 7x^2 \\
&\qquad\quad - 18x - 9 \leq 0 \wedge z - 1 \leq 0 \big] \\
&\quad \vee [y-1 \leq 0 \wedge z-1 \leq 0] \Big].
\end{aligned} \tag{4}
$$

How



Conclusions

The more I learn, the less I know, but

- There's more than one way to state a problem
- Clearly equivalent in terms of **decidability**, but not **practical computability**
- The differences are vast in practice
- We have some reasonable heuristics
- But much more work needs to be done, theoretically, experimentally, and on the "software packaging" side
- We need practical work on alternative methods for quantifier elimination

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