The Complexity of Quantifier Elimination and Cylindrical Algebraic Decomposition

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The big picture

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Our paper is about ...

- a **problem** — real quantifier elimination (QE), and
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- a **geometric object** – cylindrical algebraic decomposition (CAD).

It presents proofs ...

- that in the worst case, the problem of real QE is **very hard** $\longleftarrow$ not new
- that in the worst case, CADs are **very big** even for QE problems that are not **very hard** $\longleftarrow$ new!
- that "variable ordering" can make the difference between **very hard** and **very easy** CAD construction problems in some cases, while in others all orderings lead to **very hard** CAD construction problems $\longleftarrow$ new!

Talk Outline

1. Define QE problem

2. Describe result complexity of QE

3. Describe CAD

4. Describe result on complexity (size) of CAD

- **Satisfiability of polynomial systems**
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- **Quantifier elimination**
  Given a quantified Tarski formula with parameters, find a Tarski formula defining necessary and sufficient conditions on the parameters for the satisfiability of the input formula.

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$$\exists R \forall x, y [P_s(x, y) = 0 \Rightarrow x^2 + y^2 < R^2] \iff s \leq -1 \vee s > 1/3$$

The Complexity of Quantifier Elimination

- Davenport-Heinz (1988)
  - Family of non-linear formulas, n variables, 2 parameters
  - Any equivalent formula has length $\Omega(2^{2^{n/5}})$ assuming dense representation

- Weispfenning (1988) — Based on a construction from Fischer-Rabin (1974)
  - Family of linear formulas in n quantified variables, 1 parameter
  - Any equivalent formula has length $\Omega(2^{2^{n/5}})$ assuming each equality/inequality is linear.

- Our result
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  - Any equivalent formula has length $\Omega(2^{2^{n/3}})$ assuming ...

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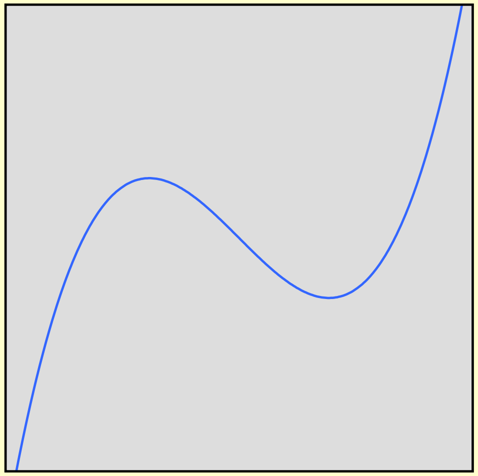

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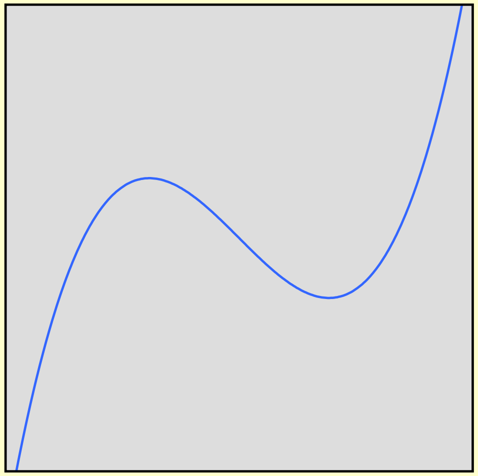

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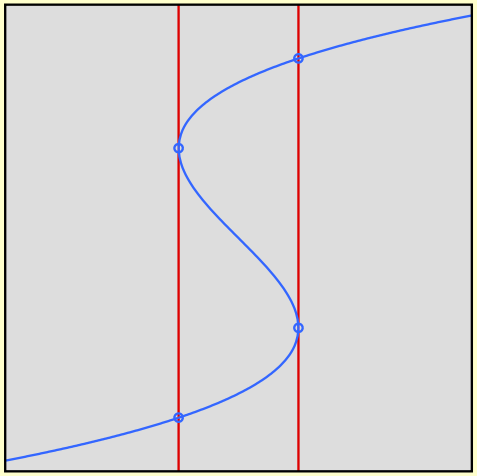

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  Implies worst case is $\Omega\left(2^{2^{\sqrt{2/3n}}}\right)$, unconstrained variable order $\leftarrow$ new

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- Variable order in CAD *can be* crucially important.
- There is a true gap between CAD-based QE and several more modern QE algorithms on QE problems with few alternations.