Lessons between Computer Algebra and Verification/Satisfiability Checking

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But first, a word from our sponsors

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A Personal Reflection

Q Are you a happy computer science professor?

JHD Yes: several times a week I put my life in the hands of my ex-students, and I am happy with this!

Q *How* does this happen?

JHD Several work for a local software house, writing railway signalling and air traffic control software

Q When mine write code, it has bugs!

JHD Same, but they don't *deliver* bugs

Q How come?

JHD Program *verification*, based on *satisfiability*.

JHD For example National Air Traffic System has 1,000,000 crash-free hours

\+ Métro ligne 14 (driverless) software delivered in 1999: no bug reports

History: Computer Algebra

- **1894** Bachmann [Bac94] invents O-notation.
- **1953** First MSc theses in Computer Algebra
- **1961** Slagle's AI thesis [Sla61] "integrates better than a freshman".
- **1966** First Computer Algebra Conference (SYMSAC)
- **1967** Moses' thesis [Mos67], beating [Sla61] algorithmically, moves computer algebra out of AI
- **1974** Knuth [Knu74] popularises O etc. in computer science
- **today** Annual ISSAC conferences, dominated by complexity results
- **And** ACA and other conferences.

History: SAT Solving

- 1971 Cook [Coo71] shows that 3-SAT is NP-complete.
- 1988 Exponential lower bounds for resolution (DPLL) solvers
- 1993 Modern SMT (Satisfiability Modulo Theories) starts [AG93]
- ∼1995 CDCL (Conflict Driven Clause Learning) introduced
- 1996 First SAT conference
- 2001 "Two watched literals" invented [MMZ⁺01]

  "just" a programming hack, but powerful
- 2003 First SMT[2] workshop
- today Annual SAT conferences and SMT workshops, with contests a major feature

[2] Then known as PDPAR.

Reminder of SAT

Problem (SAT)

Given a Boolean formula (in CNF) ($l_{ij} \in \{x_k, \overline{x_k}\} : 1 \leq k \leq m$)

$$(l_{11} \vee l_{12} \vee \cdots) \wedge (l_{21} \vee l_2 \vee \cdots) \wedge \cdots \wedge (l_{n,1} \vee l_{n,2} \vee \cdots) \qquad (1)$$

find values of $x_k \in \{T, F\}$ *to make (1) true, or return UNSAT*

These days, contests ask for an "UNSAT core", i.e. a (locally) minimal unsatisfiable equivalent.
NB: the global minimum is impracticable [CGS11, §3.1].
SAT examples with $m, n > 10^6$ occur routinely in hardware verification, and are routinely solved.

Statement of SMT

Problem (SMT)

Given a Boolean formula (possibly in CNF) ($l_{ij} \in T$ a theory)

$$(l_{11} \vee l_{12} \vee \cdots) \wedge (l_{21} \vee l_2 \vee \cdots) \wedge \cdots \wedge (l_{n,1} \vee l_{n,2} \vee \cdots) \quad (2)$$

find values in the theory to make (2) true, or return UNSAT (possibly also an UNSAT core)

There are many possible theories: SMT-LIB `http://smtlib.cs.uiowa.edu/theories.shtml` lists seven, such as `Reals`, But there are over 50 "benchmark categories", such as `QF_NRA` (quantifier-free nonlinear real arithmetic[3]).

[3] SMT-speak uses "arithmetic" where this community would use "algebra".

Statement of SMT QF_NRA

Problem (SMT–QF_NRA)

Given a Boolean formula (possibly in CNF) ($l_{ij} := f_{ij}\sigma_{ij}0$, $f_{ij} \in \mathbf{Q}[x_1, \ldots, x_k]$, $\sigma_{ij} \in \{=, \neq, <, >, \leq, \geq\}$)

$$(l_{11} \vee l_{12} \vee \cdots) \wedge (l_{21} \vee l_2 \vee \cdots) \wedge \cdots \wedge (l_{n,1} \vee l_{n,2} \vee \cdots) \tag{3}$$

find values for $x_1, \ldots, x_k \in \overline{\mathbf{Q}}$ to make (3) true, or return UNSAT.

Fortunately $\overline{\mathbf{Q}}$ is sufficient, but $\mathbf{Q}$ is not ($x^2 - 2 = 0$).
We could ask for an UNSAT core here as well, but one tends to need an "UNSAT core+proof", a concept that's still not well-defined.

Translation into Computer Algebra

Problem

Solve (or prove insoluble)

$$\exists x_1 \cdots \exists x_k \Phi(f_i \sigma_i 0) : \tag{4}$$

$f_i \in \mathbf{Q}[x_1, \ldots, x_k]$, $\sigma_i \in \{=, \neq, <, >, \leq, \geq\}$, Φ *a Boolean combination.*

This is more specific than usual quantifier elimination/Cylindrical Algebraic Decomposition, as all variables are quantified with the same quantifier, and hence the doubly-exponential bounds [BD07, DH88] don't apply.

Goals

Computer Algebra Describe the space of all solutions

Satisfiability Find one solution, or UNSAT

#SAT is the problem of counting all solutions, and this is known to be much harder in practice

MAXSAT is the problem of finding the "best" solution, also much harder in practice

worse Cylindrical Algebraic Decomposition will find all the geometry of *all* the polynomials, so will struggle with

$$(x < -1) \wedge (x > 1) \wedge \Phi(\text{big polynomials}) \tag{5}$$

whereas any decent SMT will say UNSAT immediately

Strategies

Computer Algebra:

- [Col75] Look at the f_i first
- [McC99] If Φ is $f_1 = 0 \wedge \Phi'$ process f_1 and x_k specially, then look at the $\mathrm{res}_{x_k}(f_1, f_i)$ first
- [BDE+16] Handle $(f_1 = 0 \wedge \Phi') \vee \Phi''$ as well
- [McC01] handle multiple $f_i = 0$
- [ED16] improve on this, *provided* f_i etc. are primitive

Satisfiability Modulo Theories:

- all look at the logic first
- [JdM12] Use CAD-inspired techniques to construct a refutation

[Bro13, Bro15] Feed these ideas back into Computer Algebra

- Idea An imprimitive polynomial $f(x) = 0$ is a disjunction $\mathrm{cont}(f) = 0 \vee \mathrm{pp}(f) = 0$

Other theories

- **QF_LRA** Can make use of linear programming, another field where practice is far better than theory
- **Linearise** Work in [CGI+17, Irf18] linearises multiplication, and even transcendental functions
- **Used** to verify aircraft wheel systems: note the **R** refers to the real world
- **QF_FLOAT** to verify programs *manipulating* floating-point numbers

  Tends to be done by converting into bit-vectors: very expensive.

Meta-Strategies

Computer Algebra:

Complexity	What's the worst case?
Also	Can we prove lower bounds (e.g. [BD07])
Algorithms	probably want four examples to show we're faster than some other guy
But	we run these tests, so probably comparing my experimental with his old production

Satisfiability Modulo Theories:

Benchmarks	are everything, the more examples (preferably thousands) the better
Contests	with independent jury/setting implement these.
Therefore	doing well on easy cases also matters
And	heuristics matter

But how do you present results of benchmarks on thousands of examples? [BDG17]

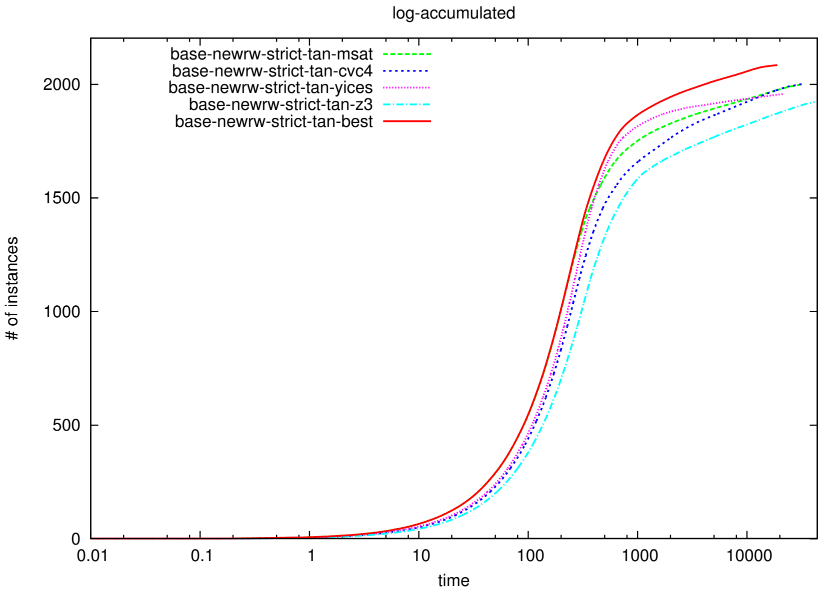
“Cactus” or “Survival” plots

The methodology for producing these, given a large benchmark set of problems, is as follows.

1. For each method separately
   1. Solve each problem p_i, noting the time t_i (up to some threshold T).
   2. Sort the t_i into increasing order (discarding the time-out ones).
   3. Plot the points $(t_1, 1)$, $(t_1 + t_2, 2)$ etc., and in general $(\sum_{i=1}^{k} t_i, k)$.
2. Place all the plots on the same axes, optionally using a logarithmic scale for time.
3. Optionally add “virtual best solver”

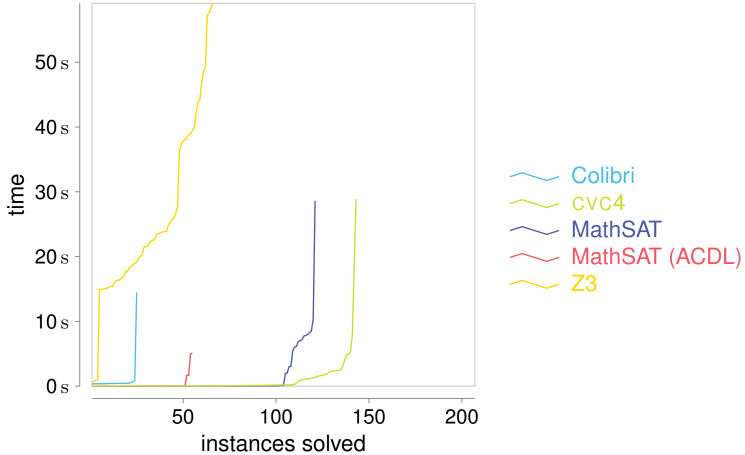
N.B. There is therefore no guarantee that the same problems were used to produce time results from different solvers.

A plot


log-accumulated
base-newrw-strict-tan-msat
base-newrw-strict-tan-cvc4
base-newrw-strict-tan-yices
base-newrw-strict-tan-z3
base-newrw-strict-tan-best
# of instances
2000
1500
1000
500
0
0.01
0.1
1
10
100
1000
10000
time

Cactus plot

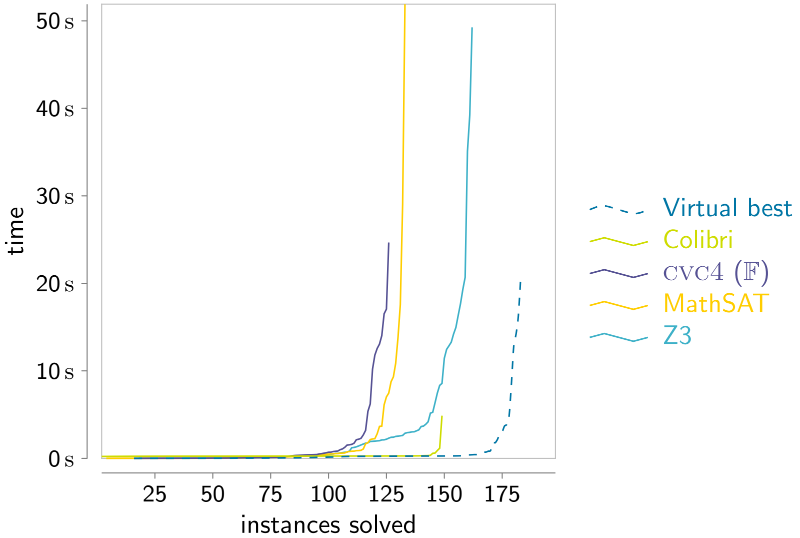
Heizmann



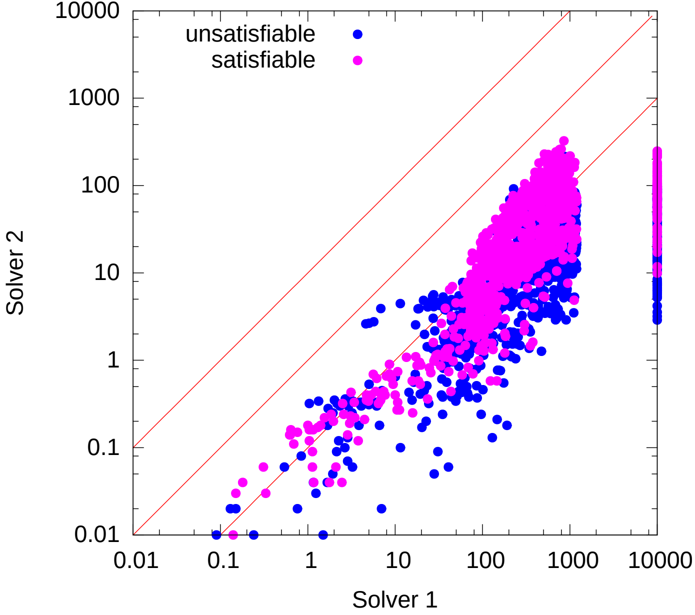
23 CVC4 IEEE-754 implementation

Plot with Virtual Best Solver

Figure: Schanda Cactus Plot



How do two solvers compare?



Conclusions

"People like theorems because they are neat, but people use software because it solves problems"

1. Computer Algebra does not make enough (?any) use of SAT solvers
2. The same is probably true of linear programming
3. Can computer algebra help with `QF_FLOAT`?
4. "If we could use reals rather than booleans for signalling, we could get 30% more trains on our tracks" SC^2 colleague

The challenge isn't writing the software — it's proving it correct, and I want to stay a happy CS professor!

??? Is ACA the right place to host benchmarking for computer algebra?

But Where do we get thousands of problems from?

By being industrially relevant, which requires demonstrating on benchmarks of thousands of problems, which . . .

SMT-Lib has similar challenges: Let's start! [WBD12]

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