On Kahan's Rules for Determining Branch Cuts

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What is a "Function" and how do we evaluate it?

Bourbaki A left-total, right-unique relation

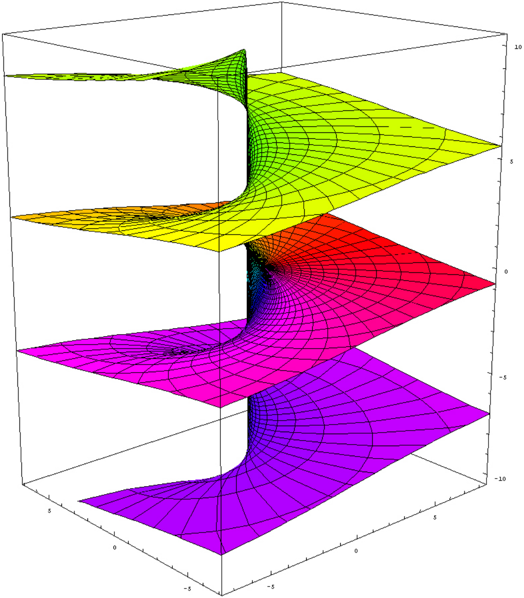
o.d.e. analytic continuation from initial conditions

inverse analytic continuation from initial point

differential algebra "what do you mean, evaluate?"

But these are incompatible: $\log \stackrel{?}{=} \int \frac{1}{x} \stackrel{?}{=} \exp^{-1} \stackrel{?}{=}$ "$\theta : \theta' = \frac{1}{x}$".

An example: log



Various solutions

- Multivalued Functions — Passing the buck forwards
- Riemann surfaces — Passing the buck backwards
- “in a suitably chosen open subset” — passing the buck upwards
- branch cuts — biting the bullet; sacrificing continuity for uniqueness $\mathbf{C} \to \mathbf{C}$

As well as sacrificing continuity,

We also sacrifice (some) identities

- $\log(1/z) = -\log z$ except on the negative real axis
  $\mathrm{Log}(-1) = \{(2k+1)i\pi\} = -\{(2k+1)i\pi\}$
  $\log(-1) = i\pi \neq -\log(-1)$

Riemann $1/(-1) = -1$ on a different sheet!

- $\sqrt{z-1}\sqrt{z+1)} = \sqrt{z^2-1}$ only when $\Re(z) \geq 0$

But $\sqrt{1-z^2} = \sqrt{1-z}\sqrt{1+z}$ everywhere

Hence the question is: which identities, and where?

This is not just a theorist's paradox [Kah87, pp. 187–188]

With the usual definitions,

$$g(z) := 2\operatorname{arccosh}\left(1+\frac{2z}{3}\right) - \operatorname{arccosh}\left(\frac{5z+12}{3(z+4)}\right) \qquad (1)$$

is only the same as the ostensibly more efficient

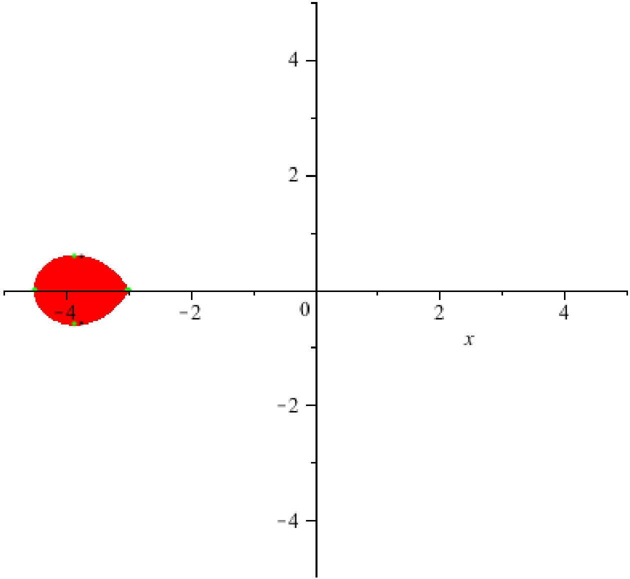
$$q(z) := 2\operatorname{arccosh}\left(2(z+3)\sqrt{\frac{z+3}{27(z+4)}}\right), \qquad (2)$$

If we avoid the negative real axis and the area

$$\left\{z = x+iy : |y| \leq \sqrt{\frac{(x+3)^2(-2x-9)}{2x+5}} \wedge -9/2 \leq x \leq -3\right\}$$

- So the cuts matter.

The erroneous region



Then why branch cuts?

The table maker's/software writer's challenge: assign a *definite meaning* to log (or arccot or $\ldots$) as a function $\mathbf{C} \rightarrow \mathbf{C}$.
If we are to implement a library of functions, they must have a *context-free* definition.
What do we put in our catalogue, and what relationship will exist between the objects in the catalogue

Two questions

positioning Where do we put the cuts?

adherence What are the values on the cuts themselves?

"There can be no dispute about where to put the slits; their locations are deducible. However, Principal Values have too often been left ambiguous on the slits." [Kah87]

* Note that Kahan advocated "signed zeros", so his cuts could adhere to both sides:

$$\frac{1}{1+0i} = 1 - 0i$$

Making a table/dictionary/library/. . .

History Chosen by the author(s)

* Occasionally incompatible (Matlab in 2000!)

A+S/DLMF `http://dlmf.nist.gov` A serious attempt to systematize history, based on careful hand editing

DDMF `http://ddmf.msr-inria.inria.fr` A serious attempt to *automate* the analysis of special functions

Kahan's rules for the table maker

- R1 These functions f are extensions to **C** of a real elementary function analytic at every interior point of its domain, which is a segment $\mathcal{S}$ of the real axis.
- R2 Therefore, to preserve this analyticity (i.e. the convergence of the power series), the slits cannot intersect the interior of $\mathcal{S}$.
- R3 Since the power series for f has real coefficients, $f(\overline{z}) = \overline{f(z)}$ in a complex neighbourhood of the segment's interior, so this should extend throughout the range of definition. So complex conjugation should map slits to themselves.
- R4 Similarly, the slits of an odd function should be invariant under reflection in the origin, i.e. $z \rightarrow -z$.
- R5 The slits must begin and end at singularities.

While these rules are satisfied by the branch cuts of elementary building blocks [DLMF], we must add a form of Occam's razor:

- R6. The slits might as well be straight lines.

Wasn't there an issue about arccot?

Indeed so: between the first and ninth printings of [AS64]

$$\mathrm{arccot}_1(x) = \pi/2 - \arctan(x),$$
$$\mathrm{arccot}_9(x) = \arctan(1/x).$$

$\mathrm{arccot}_1(x)$ is defined on $\mathcal{S} = (-\infty, \infty)$, but $\mathrm{arccot}_9(x)$ on $\mathcal{S} = (0, \infty] \overset{\text{cts}}{\cup} [-\infty, 0)$, hence

R1 These functions f are extensions to **C** of a real elementary function analytic at every interior point of its domain, which is a segment $\mathcal{S}$ of the real axis.

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work differently

We extend Kahan's rules to o.d.e.s $L(y) = 0$ & initial value

- R2' The branch cuts do not enter the circle of convergence (with respect to the initial value).
- R3' Complex conjugation is respected.
- R4' Any symmetries inherent in the power series are respected.
- R5' The branch cuts begin and end at singularities.
- R6 The branch cuts are straight lines.
- * Compatible with cuts in [DLMF]

These rules *do not* necessarily completely determine the branch cut: a "random" differential equation with singularities scattered in the complex plane and no special symmetries will not be determined.

An example: $x(1+x^4)f''+(3x^4-1)f'=0$; $f(0)=f'(0)=0; f''(0)=2$

The equation has four regular singularities at $z=\pm\sqrt{\pm i}=\frac{\pm 1\pm i}{\sqrt{2}}$

(R6) These four singularities have to be connected by straight lines.

(R2′) We cannot connect the singularities pairwise (in either way!) without going to infinity.

(R4′) The symmetry $f(ix)=-f(x)$ can be checked directly from the equation, so that branch cuts should be mapped to branch cuts by a rotation of $\pi/2$.

(R3′) Reality implies that branch cuts are also mapped to branch cuts by horizontal symmetry.

We are thus left with only the following choice: cuts that "head northeast" from $\frac{1+i}{\sqrt{2}}$, "northwest" from $\frac{-1+i}{\sqrt{2}}$ etc., all meeting at infinity. This is consistent with $\arctan(x^2)$, a solution of L.

Series?

In the world of (generalized) power series, the branch cuts

- Only appear in the expansions about the singular points
- Their directions are coded in the arguments

a) $\log z =$ branch cut heading west, adhering north

b) $\log(iz) =$ branch cut heading north, etc.

- Their adherence is coded similarly

c) $-\log(1/z) =$ branch cut west, adhering south

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