Numerical Analysis for high-frequency Helmholtz problems

Four lectures by Ivan Graham, University of Bath, UK
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In these lectures I will describe recent and current research - as well as some interesting open problems - in the general area of frequency domain wave propagation and scattering. I will focus on scalar waves described by the Helmholtz equation:

$$-(\Delta + k^2)u = f, \tag{1}$$

posed in some domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, although the students will be encouraged to read some papers on the Maxwell system as well. This equation arises from applying the Fourier transform to the linear wave equation, in which case the wavenumber k takes the form $k = \omega/c$, where ω is the angular frequency and c is the wave speed.

In exterior scattering problems Ω is the exterior of a bounded scatterer and u typically satisfies a boundary condition on the scatterer boundary, together with the Sommerfeld radiation condition at infinity. When the wavenumber k is constant and the source f vanishes, boundary element methods are suitable for solving this problem, having the distinct advantage that the problem over the infinite domain is reduced to the problem over the bounded scatterer. For more general heterogeneous problems, domain methods such as finite elements are normally used. In this context a suitable first model problem is the interior impedance problem

$$\begin{aligned} -(\Delta + k^2)u &= f \quad \text{in } \Omega, && (2)\\ \frac{\partial u}{\partial n} - \mathrm{i}ku &= g \quad \text{on } \Gamma, && (3) \end{aligned}$$

where Ω is now a bounded domain, Γ represents an artificial boundary modelling the far field, and the impedance boundary condition (3) is an approximation to the decay condition at infinity. More advanced treatment of the artificial boundary can be done through the introduction of an absorbing layer (the "PML").

Recently there has been much interest in the solution of these problems in the case when ω (and hence k) is large. Typically the solution u exhibits high oscillation and the number of degrees of freedom needed by conventional finite or boundary element methods grows at least like $\mathcal{O}(\omega^d)$ where d is the dimension of the spatial domain, so that $d = 3$ (resp. 2) for finite elements (resp. boundary elements) in 3D. For low order finite elements, the degrees of freedom needed can grow even faster than this, due to appearance of the "pollution effect". The rigorous numerical analysis of these problems in the high frequency case requires careful frequency explicit estimates for the relevant operators and approximation processes.

There are (at least two) possible approaches to the high frequency problem and we will discuss both of these in the lectures:

A1 Look for new methods with complexity which grows more slowly as the frequency increases, using knowledge of high-frequency asymptotics;

A2 Look for faster implementations of conventional methods (e.g. fast solvers), reducing the disadvantage of the high complexity

Regarding A1, one focus has been the introduction of new methods which replace the traditional piecewise polynomial methods with bases which better model the oscillatory behaviour. Such methods typically introduce oscillatory basis functions such as plane waves locally into the bases. The numerical analysis of these "Hybrid Numerical Asymptotic" (HNA) methods is both challenging and rewarding and for some problems it is even possible to obtain uniform accuracy without increasing the number of degrees of freedom as $\omega \to \infty$, although the geometry of the domain has to be encoded in the method to achieve this.

Regarding A2, there is a substantial challenge and a lot of recent interest in the construction of fast solvers which are (as) robust (as possible) as the frequency increases. The theory requires working with convergence exstimates for general Krylov methods and extending analytic settings for elliptic PDEs to the indefinite Helmholtz case. The student will meet recent rigorous results on domain decomposition together with a range of promising new methods which are currently lacking a substantial analysis.

In this sequence of lectures I am going to describe recent work in this general area. Presentations will cover some of the publications listed below, together with other works of other authors to be added. Presentations will use a mix of beamer and blackboard exposition. More detailed abstracts of the lectures will be provided.

References

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