On domain decomposition preconditioners for finite element approximations of the Helmholtz equation using absorption

Ivan Graham and Euan Spence (Bath, UK)

Collaborations with:

Paul Childs (Emerson Roxar, Oxford),
Martin Gander (Geneva)
Douglas Shanks (Bath)
Eero Vainikko (Tartu, Estonia)

CUHK Lecture 3, January 2016

Outline of talk:

- Seismic inversion, HF Helmholtz equation
- FE discretization, preconditioned GMRES solvers
- sharp analysis of preconditioners based on absorption
- new theory for Domain Decomposition for Helmholtz
- almost optimal (scalable) solvers (2D implementation)
- some open theoretical questions

Marine seismic

Seismic inversion

Inverse problem: reconstruct material properties of subsurface (characterised by wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \;=\; f \quad \text{or its elastic variant}$$

Frequency domain:

$$-\Delta u - \left(\frac{\omega}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.

Seismic inversion

Inverse problem: reconstruct material properties of subsurface (wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{\partial^2 u}{\partial t^2} = f \quad \text{or its elastic variant}$$

Frequency domain:

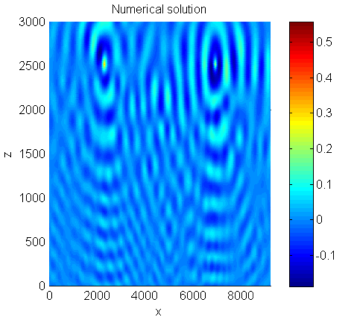
$$-\Delta u - \left(\frac{\omega L}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

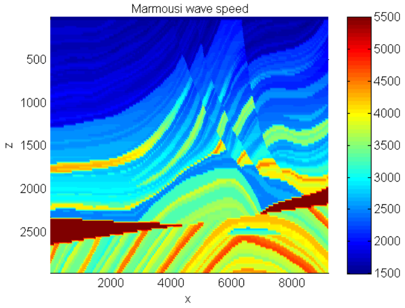
solve for u with approximate c.

Large domain of characteristic length L.

effectively high frequency - time domain vs freqency domain

Marmousi Model Problem



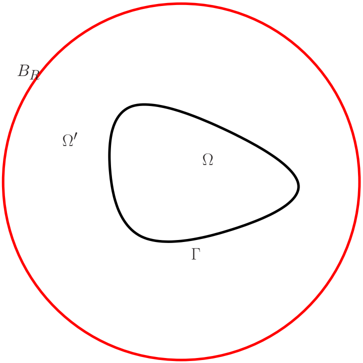


- Schlumberger 2007: Solver of choice based on principle of limited absorption (Erlangga, Osterlee, Vuik, 2004)
- This work: Analysis of this approach and use it to build better methods .....

Analysis for: interior impedance problem

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in} \quad \text{bounded domain} \quad \Omega \\ \frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \Gamma := \partial\Omega \end{aligned}$$

....Also truncated sound-soft scattering problems in Ω'



Linear algebra problem

- weak form ,

$$a\ (u,v) \ := \int_\Omega (\nabla u . \nabla \overline{v} - \quad k^2\ u\overline{v}) - \mathrm{i}k \int_\Gamma u\overline{v}$$

$$= \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}$$

- (Fixed order) finite element discretization

$$\mathbf{A}\ \ \mathbf{u}\ := (\mathbf{S} - \quad k^2\ \mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

Often: $h \sim k^{-1}$ **but pollution effect:**
for quasioptimality need $h \sim k^{-2}$?? , $h \sim k^{-3/2}$??

Melenk and Sauter 2011, Zhu and Wu 2013

Linear algebra problem

- weak form **with absorption** $k^2 \to k^2 + \mathrm{i}\varepsilon$,

$$a_\varepsilon(u,v) := \int_\Omega \left(\nabla u.\nabla \overline{v} - (k^2 + i\varepsilon)u\overline{v}\right) - \mathrm{i}k \int_\Gamma u\overline{v}$$
$$= \int_\Omega f\overline{v} + \int_\Gamma g\overline{v} \quad \text{“Shifted Laplacian''}$$

- Finite element discretization

$$\mathbf{A}_\varepsilon \mathbf{u} := (\mathbf{S} - (k^2 + i\varepsilon)\mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

Blackboard

Preconditioning with A_ε^{-1} and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

"Elman theory" for GMRES requires:

$$\|\mathbf{A}_\varepsilon^{-1}\mathbf{A}\| \lesssim 1, \quad \text{and} \quad \mathrm{dist}(0, \mathbf{fov}(\mathbf{A}_\varepsilon^{-1}\mathbf{A})) \gtrsim 1$$

Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$. **Blackboard**

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f}, \quad \text{where} \quad \mathbf{B}_\varepsilon^{-1} \approx \mathbf{A}_\varepsilon^{-1}.$$

Writing

$$\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A} = \mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon + \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon(\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}),$$

a sufficient condition is:

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \ \text{and} \ \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \ \text{small},$$

i.e. $\mathbf{A}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}$
and $\mathbf{B}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}_\varepsilon$.

Preconditioning with $\mathbf{A}_\varepsilon^{-1}$ and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

"Elman theory" for GMRES requires:

$$\|\mathbf{A}_\varepsilon^{-1}\mathbf{A}\| \lesssim 1, \quad \text{and} \quad \mathrm{dist}(0, \mathbf{fov}(\mathbf{A}_\varepsilon^{-1}\mathbf{A})) \gtrsim 1$$

Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f},$$

$\mathbf{B}_\varepsilon^{-1}$ easily computed approximation of $\mathbf{A}_\varepsilon^{-1}$. Writing

$$\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A} = \mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon + \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon(\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}),$$

so we require

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \text{ and } \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \quad \text{small},$$

i.e. **$\mathbf{A}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}$**
and $\mathbf{B}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}_\varepsilon$. **Part 1**

Preconditioning with A_ε^{-1} and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

"Elman theory" for GMRES requires:

$$\|\mathbf{A}_\varepsilon^{-1}\mathbf{A}\| \lesssim 1, \quad \text{and} \quad \mathrm{dist}(0, \mathbf{fov}(\mathbf{A}_\varepsilon^{-1}\mathbf{A})) \gtrsim 1$$

Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f},$$

B_ε^{-1} easily computed approximation of A_ε^{-1}. Writing

$$\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A} = \mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon + \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon(\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}),$$

so we require

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \ \text{and} \ \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \quad \text{small},$$

i.e. A_ε^{-1} **to be a good preconditioner for** A
and B_ε^{-1} **to be a good preconditioner for** A_ε **. Part 2**

A very short history

Bayliss et al 1983 , Laird & Giles 2002.....

Erlangga, Vuik & Oosterlee '04 and subsequent papers:
Precondition $\mathbf{A}$ with MG approximation of $\mathbf{A}_\epsilon^{-1}$ $\epsilon \sim k^2$
(simplified Fourier eigenvalue analysis)

Kimn & Sarkis '13 used $\varepsilon \sim k^2$ to enhance domain decomposition methods

Engquist and Ying, '11 Used $\varepsilon \sim k$ to stabilise their sweeping preconditioner

...others...

Theorem 1 (with Martin Gander and Euan Spence)
For star-shaped domains
Smooth (or convex) domains, quasiuniform meshes:

$$\|\mathbf{I} - \mathbf{A}_\epsilon^{-1}\mathbf{A}\| \quad \lesssim \quad \frac{\epsilon}{k} \, .$$

Corner singularities, locally refined meshes:

$$\|\mathbf{I} - \mathbf{D}^{1/2}\mathbf{A}_\epsilon^{-1}\mathbf{A}\mathbf{D}^{-1/2}\| \quad \lesssim \quad \frac{\epsilon}{k} \, .$$

$\mathbf{D} = \mathrm{diag}(\mathbf{M}^\Omega)$.

So ϵ/k sufficiently small $\implies$ $k-$independent GMRES convergence.

Shifted Laplacian preconditioner $\varepsilon = k$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**6**
$h \sim k^{-3/2}$	20	**6**
	40	**6**
	80	**6**

Shifted Laplacian preconditioner $\varepsilon = k^{3/2}$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

$h \sim k^{-3/2}$

k	# GMRES
10	**8**
20	**11**
40	**14**
80	**16**

Shifted Laplacian preconditioner $\varepsilon = k^2$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**13**
$h \sim k^{-3/2}$	20	**24**
	40	**48**
	80	**86**

Proof of Theorem 1: via continuous problem

$$a_\epsilon(u,v) = \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}\,, \quad v \in H^1(\Omega) \qquad (*)$$

Theorem (Stability) Assume Ω is Lipschitz and star-shaped. Then, if ϵ/k sufficiently small,

$$\underbrace{\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)}}_{=:\|u\|^2_{1,k}} \lesssim \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)}\,, \quad k \to \infty$$

"$\lesssim$" indept of k and ϵ cf. Melenk 95, Cummings & Feng 06

More absorption: $k \lesssim \epsilon \lesssim k^2$ general Lipschitz domain OK.

Key technique in proof: **Rellich/Morawetz Identities**

More detail of proof: Lecture 4

Proof continued

Exact solution estimate :

$$\|u\|_{L^2(\Omega)} \lesssim k^{-1}\|f\|_{L^2(\Omega)} \qquad (*)$$

Finite element solution: $\mathbf{A}_\varepsilon \mathbf{u} = \mathbf{f}$

Estimate:

$$\|\mathbf{u}\|_2 \lesssim k^{-1}h^{-d}\|\mathbf{f}\|_2 \qquad (**)$$

proof of (**) uses (*) and FE quasioptimality
(h small enough) **Lecture 4**

Hence
$$\|\mathbf{I} - \mathbf{A}_\epsilon^{-1}\mathbf{A}\| \le \|\mathbf{A}_\varepsilon^{-1}\|\|\mathbf{A}_\varepsilon - \mathbf{A}\| \le k^{-1}h^{-d}\,\|i\epsilon\mathbf{M}^\Omega\| \lesssim \frac{\epsilon}{k}.$$

Locally refined meshes:

$$\|\mathbf{I} - \mathbf{D}^{1/2}\mathbf{A}_\epsilon^{-1}\mathbf{A}\mathbf{D}^{-1/2}\| \lesssim \frac{\epsilon}{k}.$$

Exterior scattering problem with refinement

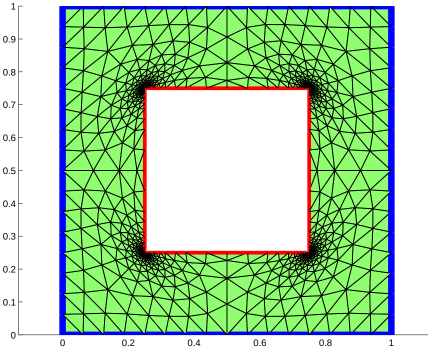
$h \sim k^{-1}$,

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

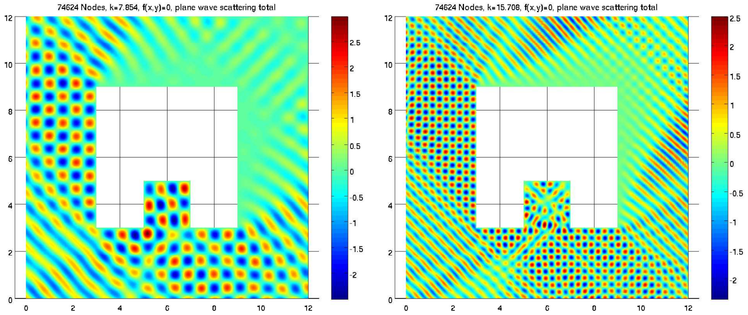
 # GMRES

with diagonal scaling

k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
20	**5**	**8**
40	**5**	**11**
80	**5**	**13**
160	**5**	**16**



A trapping domain



k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
$10\pi/8$	**18**	**29**
$20\pi/8$	**19**	**41**
$40\pi/8$	**21**	**60**
$80\pi/8$	**22**	**89**

Stability result fails ... “quasimodes” **Lecture 4**

Betcke, Chandler-Wilde, IGG, Langdon, Lindner, 2010

Part 2: How to approximate A_ε^{-1}?

Erlangga, Osterlee, Vuik (2004):
Geometric multigrid: problem "elliptic"

Engquist & Ying (2012):
"Since the shifted Laplacian operator is elliptic, standard algorithms such as multigrid can be used for its inversion"

Domain Decomposition (DD):

Many non-overlapping methods ($\varepsilon = 0$)

Benamou & Després 1997.....Gander, Magoules, Nataf, Halpern, Dolean........

General issue: coarse grids, scalability?

Conjecture If ε large enough, classical overlapping DD methods with coarse grids will work (giving scalable solvers).

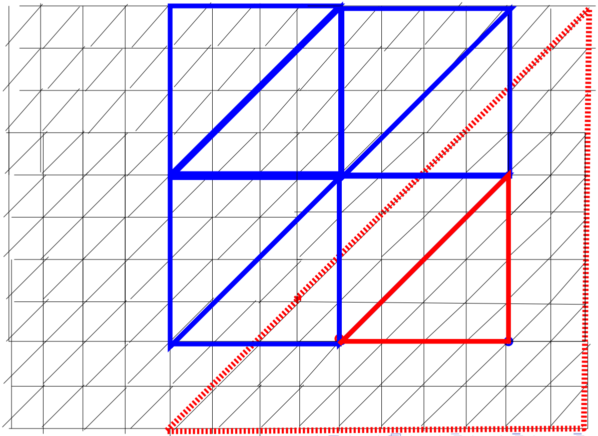
However Classical analysis for $\varepsilon = 0$ (Cai & Widlund, 1992) leads to coarse grid size $H \sim k^{-2}$

Classical additive Schwarz

To solve a problem on a fine grid FE space $\mathcal{S}_h$

- **Coarse space $\mathcal{S}_H$** (here linear FE) **on a coarse grid**
- **Subdomain spaces $\mathcal{S}_i$ on subdomains Ω_i**, overlap δ

$H_{sub} \sim H$ in this case



Classical additive Schwarz p/c for matrix C

Approximation of C^{-1}:

$$\sum_i \mathbf{R}_i^T \mathbf{C}_i^{-1} \mathbf{R}_i \ + \ \mathbf{R}_H^T \mathbf{C}_H^{-1} \mathbf{R}_H$$

$\mathbf{R}_i$ = restriction to $\mathcal{S}_i$,
$\mathbf{C}_i = \mathbf{R}_i \mathbf{C} \mathbf{R}_i^T$
Dirichlet BCs

$\mathbf{R}_H$ = restriction to $\mathcal{S}_H$
$\mathbf{C}_H = \mathbf{R}_H \mathbf{C} \mathbf{R}_H^T$

Apply to A_ε to get $\mathrm{B}_\varepsilon^{-1}$

Non-standard DD theory - applied to A_ε

Coercivity Lemma There exisits $|\Theta| = 1$, with

$$\mathrm{Im}\,[\Theta a_\varepsilon(v,v)] \ \gtrsim \ \frac{\varepsilon}{k^2}\|v\|_{1,k}^2. \qquad (\star)$$

Projections onto subpaces:

$$a_\varepsilon(Q_i v_h, w_i) = a_\varepsilon(v_h, w_i), \quad v_h \in \mathcal{S}_h, \quad w_i \in \mathcal{S}_i \ .$$

Non-standard DD theory - applied to A_ε

Coercivity Lemma There exisits $|\Theta| = 1$, with

$$\mathrm{Im}\,[\Theta a_\varepsilon(v,v)] \gtrsim \frac{\varepsilon}{k^2} \underbrace{\|v\|_{1,k}^2}_{\|\nabla u\|_\Omega^2 + k^2\|u\|_\Omega^2} . \qquad (\star)$$

Lecture 4

Projections onto subpaces:

$$a_\varepsilon(Q_H v_h, w_H) = a_\varepsilon(v_h, w_H), \quad v_h \in \mathcal{S}_h, \quad w_H \in \mathcal{S}_H .$$

Guaranteed well-defined by $(\star)$.

Analysis of $\mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon$ equivalent to analysing

$$Q := \sum_i Q_i + Q_H \quad \text{operator in FE space} \quad \mathcal{S}_h .$$

Convergence results

Assume overlap $\delta \sim H$ and $\varepsilon \sim k^2$

Theorem (with Euan Spence and Eero Vainikko)

(i) For all coarse grid sizes H,

$$\|B_\varepsilon^{-1} A_\varepsilon\|_{D_k} \lesssim 1 \, .$$

(ii) Provided $Hk \lesssim 1$ (no pollution!).

$$\mathrm{dist}(0, \textbf{fov}(B_\varepsilon^{-1} A_\varepsilon)_{D_k}) \gtrsim 1,$$

Note: D_k = stiffness matrix for Helmholtz energy:

$$(u, v)_{H^1} + k^2 (u, v)_{L^2}$$

Hence $k-$independent (weighted) GMRES convergence when

$$\varepsilon \sim k^2 \quad \textbf{and} \quad Hk \lesssim 1$$

Convergence results - general ε

Assume overlap $\delta \sim H$

Theorem (with Euan Spence and Eero Vainikko)

(i) For all coarse grid sizes H,

$$\|B_\varepsilon^{-1} A_\varepsilon\|_{D_k} \lesssim k^2/\varepsilon \ .$$

(ii) Provided $Hk \lesssim (\varepsilon/k^2)^3$

$$\mathrm{dist}(0, \textbf{fov}(B_\varepsilon^{-1} A_\varepsilon)_{D_k}) \gtrsim (\varepsilon/k^2)^2,$$

Same results for right preconditioning (duality)

Extension to general overlap, and one-level Schwarz

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|^2_{1,k} + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

Second term is "small" (condition on kH)

["Galerkin Orthogonality", duality, regularity]

$$\begin{aligned} |(v_h, Qv_h)_{1,k}| &\gtrsim \sum_j \|Q_j v_h\|^2_{1,k} + \|Q_H v_h\|^2_{1,k} \\ &\gtrsim \|v_h\|^2_{1,k} \end{aligned}$$

Some steps in proof $\varepsilon \ll k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

Second term is small (condition on kH)

["Galerkin Orthogonality", duality, regularity]

$$|(v_h, Qv_h)_{1,k}| \gtrsim \sum_j \|Q_j v_h\|_{1,k}^2 + \|Q_H v_h\|_{1,k}^2$$

$$\gtrsim \left(\frac{\varepsilon}{k^2}\right)^2 \|v_h\|_{1,k}^2$$

Lecture 4

Useful Variants

HRAS:

- Multiplicative between coarse and local solves
- only add up once on regions of overlap

ImpHRAS

- impedance boundary conditions on local solves

All experiments:

unit square, $h \sim k^{-3/2}, \; n \sim k^3, \; \delta \sim H$.

Standard GMRES - minimise residual in Euclidean norm
(Theory has weights)

B_ε^{-1} as preconditioner for A_ε

$\varepsilon = k^2$

$\#$ GMRES iterates with HRAS:

k	$H \sim k^{-1}$	$H \sim k^{-0.9}$	$H \sim k^{-0.8}$
10	**8**	**8**	**8**
20	**8**	**9**	**9**
40	**9**	**10**	**10**
60	**9**	**10**	**11**
80	**9**	**10**	**11**

Scope for increasing H when $\varepsilon = k^2$

Is there scope for reducing ε ?

$\mathrm{B}_\varepsilon^{-1}$ as preconditioner for A_ε

$\varepsilon = k$

$\#$ **GMRES iterates with HRAS:**

k	$H \sim k^{-1}$	$H \sim k^{-0.9}$	$H \sim k^{-0.8}$
10	**10**	**10**	**12**
20	**11**	**14**	**18**
40	**16**	**24**	**122**
60	**22**	**40**	*
80	**30**	**61**	*

Method still "works" when $\varepsilon = k$ provided $Hk \sim 1$

The real problem: $\mathrm{B}_\varepsilon^{-1}$ as preconditioner for A

$H \sim k^{-1}$

$\#$ **GMRES iterates with HRAS:**

k	$\varepsilon = k$	$\varepsilon = k^2$ cf. Shifted Laplace
10	**11**	**19**
20	**12**	**37**
40	**18**	**63**
60	**25**	**86**
80	**33**	**110**
100	**43**	**136**

Local problems of size $k \times k$
Coarse grid problem of $k^2 \times k^2$ (dominates)

The coarse grid problem: inner iteration

problem of size $k^2 \times k^2$, with $\varepsilon \sim k$
(Hierarchical) subdomains of size $k \times k$
no inner coarse grid

$\#$ **GMRES iterates with ImpHRAS**

k	$H_{inner} \sim k^{-1}$
10	**9**
20	**14**
40	**21**
60	**30**
80	**35**
100	**39**
120	**42**
140	**46**

$\sim k^{0.3}$

The real problem: Inner outer FGMRES

$\varepsilon = k$

$\#$ **FGMRES iterates with HRAS (Inner iterations ImpHRAS)**

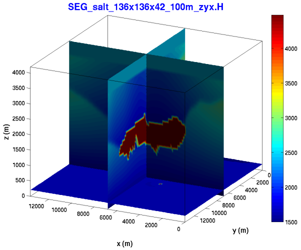
k		time (s)
10	**18 (1)**	0.66
20	**19 (2)**	3.68
40	**22 (3)**	54.7
60	**28 (5)**	370
80	**36 (5)**	1316
100	**45 (7)**	3417
		$\sim k^4 \sim n^{4/3}$

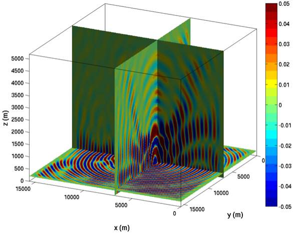
$\mathcal{O}(k^2)$ independent solves of size k

A more challenging application

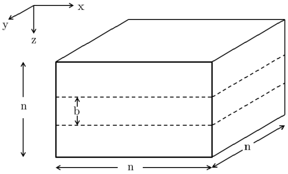
3D SEG Salt model

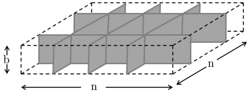
Childs, IGG, Shanks, 2016





Hybrid Sweeping preconditioner with one level **RAS** inner solve





Boundary condition chosen as “optimised Robin condition”

	$\omega = 3\pi$		$\omega = 6\pi$		$\omega = 9\pi$	
Nsub	Iterations	Solve time (s)	Iterations	Solve time (s)	Iterations	Solve time (s)
2x2x1	26	2.704e+01	29	2.995e+01	43	9.98e+01
4x4x1	26	2.470e+01	29	2.691e+01	43	9.97e+01
8x8x1	26	9.440e+00	29	1.011e+01	43	9.99e+01

PPW **24** **12** **8**

Shifted problem $(\omega/c(\mathbf{x}))^2 \to ((\omega - 1 + 0.5\mathrm{i})/c(\mathbf{x}))^2$

cf. $\varepsilon \sim k$

Summary

- k and ϵ explicit analysis allows rigorous explanation of some empirical observations and formulation of new methods.
- When $\epsilon \sim k$, $\mathbf{A}_\epsilon^{-1}$ is optimal preconditioner for $\mathbf{A}$
- When $\epsilon \sim k^2$, $\mathbf{B}_\varepsilon^{-1}$ is optimal preconditioner for $\mathbf{A}_\varepsilon$
- When preconditioning $\mathbf{A}$ with $\mathbf{B}_\varepsilon^{-1}$, empirical best choice is $\varepsilon \sim k$
- New framework for DD analysis - Helmholtz energy and sesquilinear form.
- Open questions in analysis when $\frac{\varepsilon}{k^2} \ll 1$