Lecture2:
Hybrid numerical-asymptotic methods in high-frequency scattering

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A survey of joint work with and work by:

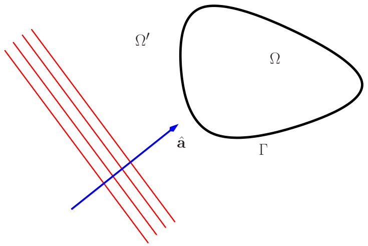
V. Domínguez (Navarra)
E.A. Spence, T.Kim (Bath),
T. Betcke, V. Smyshlyaev (Univ. College London)
S. Chandler-Wilde, S. Langdon, D. Hewitt (Reading)

CUHK, January 2016

[S.N. Chandler-Wilde, IGG, S.Langdon, E.A. Spence, Acta Numerica 21 (2012), pp 89-305]

High freq. problem for the Helmholtz equation

Given an object $\Omega \subset \mathbb{R}^d$, with boundary Γ and exterior Ω',
Incident plane wave: $u_I(x) = \exp(\mathrm{i}k\mathbf{x} \cdot \widehat{\mathbf{a}})$
wavelength $\lambda = 2\pi/k$



Total wave $u = u_I + u_S$, where **Scattered wave** u_S satisfies:

$$\Delta u_S + k^2 u_S \;=\; 0 \quad \text{in } \Omega'$$

plus **boundary condition** (Here $u_I + u_S = 0$ on Γ) and

radiation condition: $\frac{\partial u^S}{\partial r} - iku^S \;=\; o(r^{-(d-1)/2})$ as $r \to \infty$

Recap of Lecture 1

- Homogeneous scattering problem : k constant, infinite domain
- Boundary integral equation posed on scattering boundary Γ
- Solve using piecewise polynomial BEM
- Require at least $h \sim k^{-1}$ to resolve oscillations in solution

$$\implies \text{complexity } \mathcal{O}(k^{d-1})$$

- Proof that $h \sim k^{-(d+1)/2}$ is sufficient

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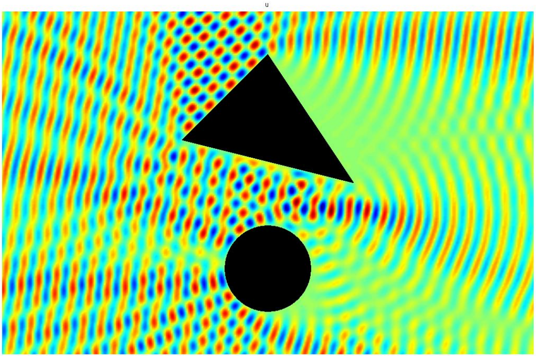
This lecture

• Different methods which have complexity (almost) bounded as $k \to \infty$

How is this possible?

A multiscale problem



Plane wave incident field $\exp(i\boldsymbol{k}\mathbf{x}.\hat{\mathbf{a}})$ scale $\mathcal{O}(\boldsymbol{k}^{-1})$.

May be other **scales in the scattered field**, $\boldsymbol{k}^{-1/2}$, $\boldsymbol{k}^{-1/3}$

Numerical Analysis

Conventional numerical methods (piecewise polynomial bases)

$\rightarrow$ at least $O(k^{d-1})$ DOF's

Conventional asymptotic methods work well as $k \rightarrow \infty$.
[Fock, Ludwig, Buslaev, Babich ....]

Today's topic: **"Hybrid numerical-asymptotic Methods"**
piecewise oscillatory bases work for all k

require $\sim \mathcal{O}(1)$ DOF's as $k \rightarrow \infty$

Need asymptotic information, so geometry dependent

Related: Plane-wave bases for general geometries

Research Plan

I. Construct oscillatory basis (for Galerkin BEM)

II. Prove error estimates

III. Realise the estimates

First formulate as BIE (last lecture)

$$\Delta u + k^2 u = 0$$

$$G_k(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{2D} \\ \dfrac{\exp(ik|x-y|)}{4\pi|x-y|} & \text{3D} \end{cases}$$

single layer potential : $(\mathcal{S}_k\phi)(x) = \int_\Gamma G_k(x,y)\phi(y)dS(y)$,

double layer: $(\mathcal{D}_k\phi)(x) = \int_\Gamma [\partial_{n(y)} G_k(x,y)]\phi(y)dS(y)$,

adjoint double layer: $\mathcal{D}'_k$ (switch roles of x and y).

Oscillatory integrals with **phase:** $k|x-y|$ **blackboard 1**

Combined potential boundary integral formulations

combined potential formulation

$$R_k v \; := \; \left(\frac{1}{2} I + \mathcal{D}'_k\right) v - \mathrm{i} k \mathcal{S}_k v \; = \; \partial_n u_I - \mathrm{i} k u_I := f_k \; ,$$

star - combined potential formulation: (requires an origin)

$$R_k v \; := \; (\mathbf{x}.\mathbf{n}) \left(\frac{1}{2} I + \mathcal{D}'_k\right) v \; + \; \mathbf{x}.(\nabla_\Gamma \mathcal{S}_k) v \; - \; \mathrm{i}\eta \mathcal{S}_k v \; = \; f_k \; ,$$

$(\mathbf{x}.\mathbf{n}) > 0$ **star-shaped**

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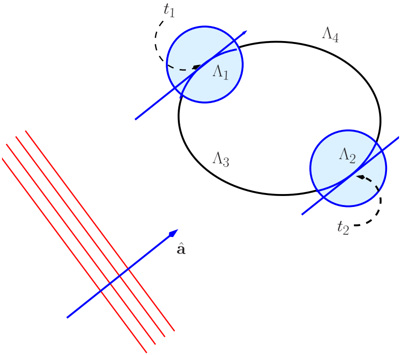
In general $\quad R_k v \;=\; f_k \quad$ No spurious frequencies.

Construct basis: 2D smooth convex case



"Physical optics" approx

$$v(\boldsymbol{\gamma}(s)) := k\ \exp(ik\boldsymbol{\gamma}(s).\widehat{\mathbf{a}})\, V(s)\ .$$

$\boldsymbol{\gamma}(s)$ = arclength

blackboard 2

V = "Slowly varying" factor in $v = \partial u/\partial n$.

- Λ_1, Λ_2: Fock zones V oscillates on scale $k^{-1/3}$

  + other complications!
- Λ_3: Illuminated V smooth, not oscillatory.
- Λ_4: Deep Shadow $V \approx 0$ exponentially

Ex: 2D smooth convex case : prove error estimate

Solve combined potential formulation with basis:

$$v_h(s) := \begin{cases} k\exp(ik\gamma(s)\cdot\widehat{\mathbf{a}})P_p(s) & \text{Illuminated zone} \\ k\exp(ik\gamma(s)\cdot\widehat{\mathbf{a}})P_p(s) & \text{Fock zones} \quad \mathcal{O}(k^{-1/3}) \\ 0 & \text{Shadow} \end{cases}$$

where $P_p =$ polynomial of degree p

Theorem (Dominguez, IGG, Smyshlyaev, 07)

$$\frac{\|v - v_h\|_{L^2(\Gamma)}}{k} \le C_n k^{1/18} \left\{ \left(\frac{k^{1/9}}{p}\right)^n + \exp(-\beta k^{\epsilon}) \right\} ,$$

for all p and $n \approx p + 1$. C_n, β are constants independent of k and $\epsilon \approx 0$.

Corollary Choosing $p \sim k^{1/9+\delta}$ "is sufficient" as $k \to \infty$.

$k-$ explicit regularity

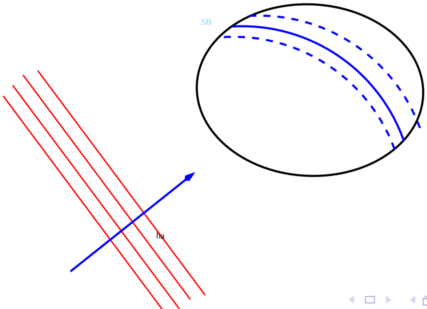
G.O. $v(\mathbf{x}) := \partial u/\partial n = kV(\mathbf{x},k)\exp(ik\mathbf{x}\cdot\hat{\mathbf{a}})\ , \quad x \in \Gamma,$

Theorem Dominguez, et. al, 2007

$$|D^n V(x,k)| \leq \begin{cases} C_n, & n = 0,1, \\ C_n\, k^{-1}\,(k^{-1/3} + \mathrm{dist}(x,SB))^{-(n+2)} & n \geq 2, \end{cases}$$

where $SB = \{\mathbf{x} \in \Gamma : \mathbf{n}(\mathbf{x}).\hat{\mathbf{a}} = 0\}$ shadow boundary.

Proof Development of Melrose and Taylor (1985) plus matched asymptotic expansions. Justifies HF Galerkin method above



Ex: 2D smooth convex case: realise the estimates

Scattering by circle

Galerkin (with quadrature - see later)

Degree of the polynomials $p_I = p_{F_1} = p_{F_2} = \mathbf{p}$

Relative error $\|v - v_h\|/k$ [**All norms** $\|\cdot\|_{L^2(\Gamma)}$]

	$k = 250$	$k = 4,000$	$k = 64,000$
$\mathbf{p} = 4$	5.57E − 03	1.57E − 03	4.69E − 04
$\mathbf{p} = 8$	6.62E − 04	2.72E − 04	7.96E − 05
$\mathbf{p} = 12$	4.43E − 04	4.55E − 05	1.50E − 05
$\mathbf{p} = 16$	1.42E − 03	3.92E − 05	6.91E − 06
$\mathbf{p} = 20$	2.47E − 03	2.74E − 04	7.43E − 06

Ex: 2D Smooth convex case: computation times

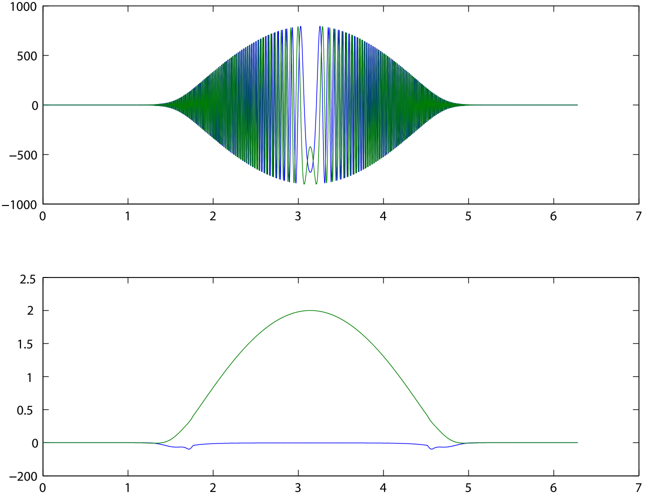
$p = 20$: 63 degrees of freedom

Times (sec) to achieve a relative error: $\leq 10^{-3}$:

k	setting up quadrature rules	assembling matrix
256	248 s	227 s
6400	227 s	230 s

Solution on circle $k = 400$

full wave solution (top)
computed slowly oscillatory part (bottom):



I. Construct oscillatory basis functions More later

II. Prove error estimates

III. Implement the methods (oscillatory integration)

1/3

II. Error Estimates: Hybrid methods

Exotic (k-dependent) subspace: $\mathcal{V}_{h,k} \subset L_2(\Gamma)$.

Galerkin method for $R_k v = f_k$:
Seek $v_h \in \mathcal{V}_{h,k}$ such that

$$(R_k v_h \ , \ w_h) = (f_k \ , \ w_h) \quad \text{for all} \quad w_h \in \mathcal{V}_{h,k}$$

Céa's lemma Assume there exist $B_k > 0, \alpha_k > 0$ such that

Continuity: $\|R_k\| \ \leq \ B_k$,
Coercivity†**:** $|(R_k v, v)| \geq \alpha_k \|v\|^2$

Then we have (with no mesh restriction),

$$\|v - v_h\| \ \leq \ \left(\frac{B_k}{\alpha_k}\right) \inf_{w_h \in \mathcal{V}_{h,k}} \|v - w_h\| \ .$$

† Stronger than invertibility.

blackboard 3

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Continuity: $\|R_k\| \ \leq \ B_k \qquad \sim k^{(d-1)/2}$ Lecture 1 ,
Coercivity†**:** $|(R_k v, v)| \geq \alpha_k \|v\|^2$ **???**

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blackboard 3

II. Some recent (positive) results - nontrapping

Combined potential formulation is **uniformly coercive** with $\alpha_k = 1/2 - \epsilon$, $\epsilon > 0$ for circle and sphere
[DoGrSm]

Fourier analysis symbol: $\frac{\pi k}{2} H^{(1)}_{|m|}(k)(J_{|m|}(k) + iJ'_{|m|}(k))$.

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blackboard 4

The star combined formulation is **uniformly coercive** $\alpha_k = \frac{1}{2}\text{ess}\inf_{\mathbf{x}\in\Gamma}(\mathbf{x}.\mathbf{n}(\mathbf{x}))$ for star-shaped Lipschitz domains.
[SpChGrSm]

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blackboard 4

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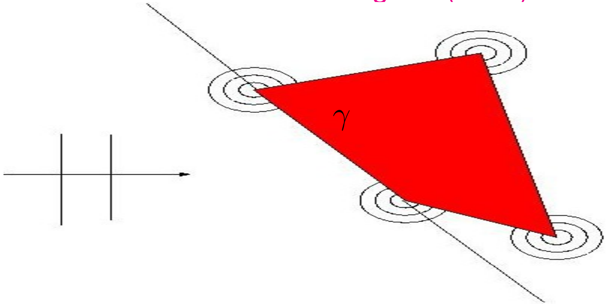
The combined potential formulation is **uniformly coercive** (for k large enough) for strictly convex smooth domains. Spence, Kamotski and Smyshlyaev, 2014

More general geometries?

convex polygon

Theorem Chandler-Wilde and Langdon (2007)



$$\frac{\partial u}{\partial n}(s) \quad = \quad 2\frac{\partial u^I}{\partial n}(s) + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)$$

where s is distance along γ, and

$$k^{-n}|v_+^{(n)}(s)| \leq \begin{cases} C_n(ks)^{-1/2-n}, & ks \geq 1, \\ C_n(ks)^{-\alpha-n}, & 0 < ks \leq 1, \end{cases}$$

where $\alpha < 1/2$ depends on the corner angle.

convex polygon - error estimate

Mesh with $\mathcal{O}(N)$ points, graded towards corners

Piecewise polynomials of degree p.

Then (under some reasonable assumption)

$$\frac{\|v - v_N\|}{k^{1/2}} \lesssim (\log(k))^{1/2} \left(\frac{\log(k)}{N} \right)^{p+1}$$

$hp-$version: Hewett, Langdon, Melenk, 2012

$$\frac{\|v - v_N\|}{k^{1/2}} \lesssim k^{\epsilon} \exp(-N^{1/2}\tau) , \quad \epsilon \in (0, 1/2), \quad \tau > 0.$$

where N is the dimension of the approximating space.

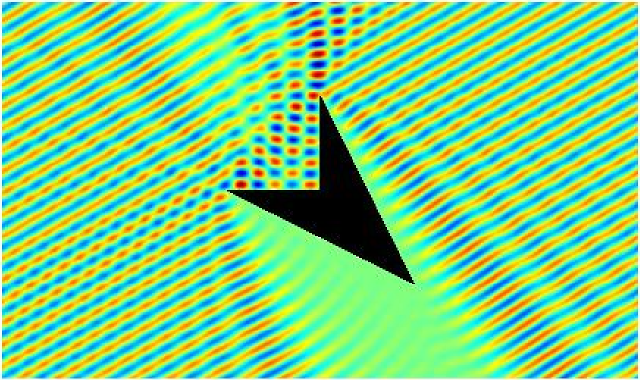
Convex Polygon

hp-scheme of Hewett, Langdon & Melenk with $N = 192$

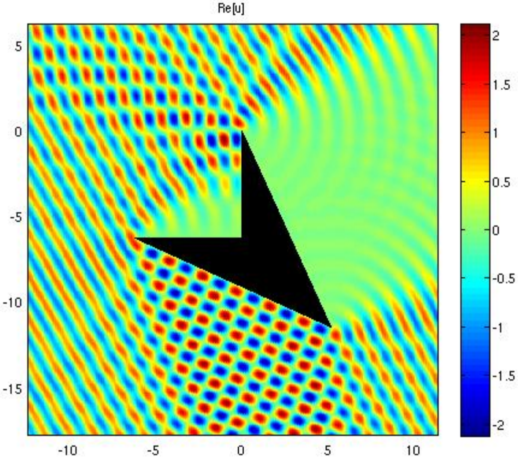
k	Relative L^2 error in $\frac{\partial u}{\partial n}$	Time (s)
10	1.46×10^{-2}	461
40	1.50×10^{-2}	615
160	1.55×10^{-2}	615
640	1.58×10^{-2}	732
2560	1.73×10^{-2}	844
10240	1.74×10^{-2}	940

Logarithmic in $\boldsymbol{k}$

non-convex polygon



non-convex polygon



Chandler-Wilde, Hewett, Langdon, Twigger, 2011:
HF Ansatz taking account of diffractions at corners and reflections

hp-BEM: Non-convex polygon
Chandler-Wilde, Hewett, Langdon, Twigger, 2011

k	dof	dof per λ	L^2 error	Relative L^2 error
5	320	10.7	2.09e-2	1.51e-2
10	320	5.3	1.07e-2	1.11e-2
20	320	2.7	4.60e-3	6.91e-3
40	320	1.3	3.13e-3	6.83e-3

Recent work from the group at Reading (UK)

S. N. Chandler-Wilde, D. P. Hewett, S. Langdon, A. Twigger, A high frequency boundary element method for scattering by a class of nonconvex obstacles, Numer. Math., 129(4), 2015

S. P. Groth, D. P. Hewett, S. Langdon, Hybrid numerical-asymptotic approximation for high frequency scattering by penetrable convex polygons, IMA J. Appl. Math., 80(2), 2015

D. P. Hewett, S. Langdon, S. N. Chandler-Wilde, A frequency-independent boundary element method for scattering by two-dimensional screens and apertures, IMA J. Numer. Anal., 35(4), 2015

D. P. Hewett, Shadow boundary effects in hybrid numerical-asymptotic methods for high frequency scattering, Euro. J. Appl. Math., 26(5), 2015

I. Construct oscillatory basis functions

II. Prove error estimates

III. Implement the methods (oscillatory integration)

III Implementing the methods: oscillatory integration

Galerkin matrix involves **oscillatory integrals**, e.g. (in 2D):

$$\int \exp(-ik\,\widehat{\mathbf{a}}.\mathbf{x})P_\ell(\mathbf{x}) \int H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\exp(ik\,\widehat{\mathbf{a}}.\mathbf{y})P_{\ell'}(\mathbf{y})dydx$$

$$= \int\int \exp(ik\{|\mathbf{x}-\mathbf{y}| + \widehat{\mathbf{a}}.(\mathbf{y}-\mathbf{x})\})M_k(\mathbf{x},\mathbf{y})\,dy\,dx$$

M_k not oscillatory. Arc-length: $\mathbf{x} = \boldsymbol{\gamma}(s),\ \mathbf{y} = \boldsymbol{\gamma}(t)$

$$\int\int \exp(ik\Psi(s,t))M_k(s,t)dtds\ ,$$

Phase: **blackboard 5**

$\Psi(s,t) = |\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(t)| + \widehat{\mathbf{a}}.(\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(s)) \ =:\ \psi_{[s]}(t)$.

Strategy: change of variable $t \to \tau$, with $\tau = \psi_{[s]}(t)$ for each s.

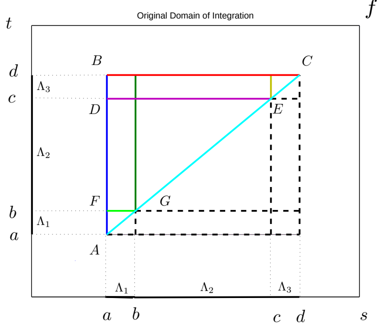
Stationary points? - Ignore for the moment

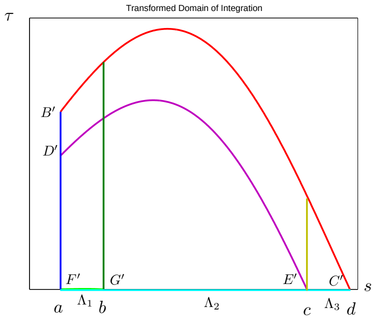
Change of variable - example

$$
\begin{aligned}
I &:= \int_b^c \int_s^c \exp(ik{\color{red}\Psi(s,t)}) M_k(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_b^c \left[\int_0^{\psi_{[s]}(c)} \exp(ik\tau) M_k(s, \psi_{[s]}^{-1}(\tau))\, |J(s,\tau)| \mathrm{d}\tau \right] \mathrm{d}s,
\end{aligned}
$$

Switching order of integration:

$$
= \int_0^{\tau_{\max}} \underbrace{\left[\int_{r_1(\tau)}^{r_2(\tau)} M_k(s, \psi_{[s]}^{-1}(\tau))\, |J(s,\tau)| \mathrm{d}s \right]}_{f(\tau)} \exp(ik\tau)\mathrm{d}\tau
$$





Filon-Censhaw-Curtis rules

$$\int_{-1}^{1} f(\tau)\exp(ik\tau)d\tau \approx \int_{-1}^{1} (Q_N f)(\tau)\exp(ik\tau)d\tau$$

Polynomial interpolant $\quad (Q_N f)(\cos(j\pi/N)) = f(\cos(j\pi/N))$
Nested, Implementation via FFT in $\mathcal{O}(N \log N)$ operations.
Stable implementation: [DoGrSm].

Theorem For $r \in [0,1]$, and all $m \geq 1$,

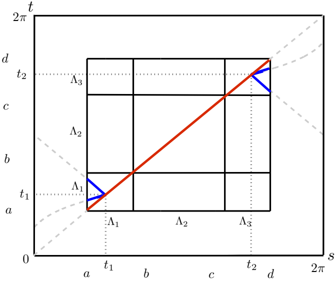
$$\left| \int_{-1}^{1} (f - Q_N f)(\tau)\exp(ik\tau)d\tau \right| \lesssim \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-r} \int_{-1}^{1} \frac{|f^{(m)}(x)|^2}{\sqrt{1-x^2}}$$

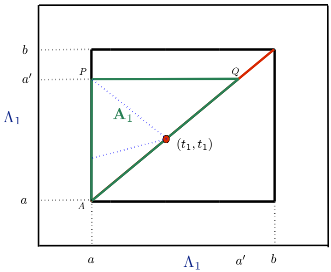
$M-$ point composite version for singularities

$$\left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \|f\|_{N+1,\text{singular}}$$

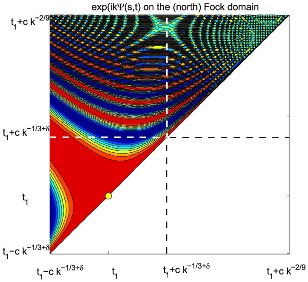
Allowing stationary points in f [DoGrKi]

Stationary points of $\psi_{[s]}$ (T. Kim, PhD)





In A_1 $\quad |D^{\mathbf{p}}_{(s,t)} \exp(ik\Psi(s,t))| \lesssim k^{|\mathbf{p}|/3}$ **Use conventional rules**



Ellipse with $a = 3$, $b = 1$. Relative errors at the point where the incident wave is orthogonal to Γ. (**T. Kim**)

p	$k = 1000$	$k = 4000$	$k = 8000$	$k = 16000$	relative time
6	$3.70(-3)$	$2.43(-2)$	$4.31(-2)$	$8.32(-2)$	1
8	$3.24(-3)$	$8.62(-3)$	$1.74(-2)$	$2.56(-2)$	1.5
10	$2.69(-3)$	$3.35(-3)$	$7.23(-3)$	$9.79(-3)$	2.3
12	$2.47(-3)$	$1.97(-3)$	$3.07(-3)$	$2.90(-3)$	3.1
14	$3.15(-3)$	$1.27(-3)$	$1.39(-3)$	$1.49(-4)$	4.1
16	$4.06(-3)$	$9.28(-4)$	$6.15(-4)$	$8.12(-5)$	5.3
18	$2.84(-3)$	$1.43(-3)$	$5.46(-4)$	$2.81(-5)$	6.8
			$\mathcal{O}(\exp(-\mathbf{0.4p}))$		$\approx \mathcal{O}(\mathbf{p^2})$

Table: $|2i - \tilde{V}_d(\pi, k)|$, $a = 3$,$b = 1$.

- In this variant computational times are fixed w.r.t. k.
- For fixed very small p, errors grow slightly with k. For larger p, errors decrease as $k \to \infty$.
- For fixed k the rate of convergence appears exponential in p and computational time is about $\mathcal{O}(p^2)$.

Summary

• Highly oscillatory scattering problem solved in time which is empirically close to $\mathcal{O}(1)$ as $k \to \infty$.

• The method and analysis are geometry dependent

So are ray tracing algorithms

• Galerkin approach and knowledge of asymptotics allow rigorous error estimates

• New results: asymptotics of solutions, estimates for oscillatory integral operators and quadrature for oscillatory integrals

• 3D presents significant challenges:
3D screen problems: [Chandler-Wilde, Langdon, Hewett, 2012, 2015]