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A class of differential-delay systems with hysteresis: Asymptotic behaviour of solutions[☆]

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Abstract

A class of differential-delay systems with hysteresis is considered. Conditions ensuring boundedness of solutions and related asymptotic and integrability properties are expressed in terms of data associated with the linear component of the overall system and a Lipschitz constant associated with the hysteretic component.

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1. Introduction

By way of motivation, consider a damped mechanical system with hysteretic restoring force and subject to delay. Depending on the nature of the delay, such a system may take one of the following two forms:

$$x''(t) + ax'(t) + bx(t) + (\Phi(x(\cdot - h)))(t) = 0,$$

or

$$x''(t) + ax'(t) + bx(t) + (\Phi(x))(t - h) = 0,$$

with parameters $a > 0$, $b \geq 0$, $h \geq 0$ and a hysteresis (*i.e.*, causal and rate-independent) operator Φ. For example, in situations where the hysteretic term is generated by the actuation of feedback signals, the former structure applies in the context of output or measurement delay, whilst the latter structure relates to the context of input or actuation delay.

As a second motivating example, consider a damped mechanical system with input hysteresis Ψ and output/measurement delay $h \geq 0$, under the feedback action of integral control with gain parameter κ and constant reference input r, as shown in Fig. 1. This system may be expressed as

$$y''(t) + ay'(t) + by(t) = (\Psi(z))(t), \quad z'(t) = \kappa(r - y(t - h)),$$

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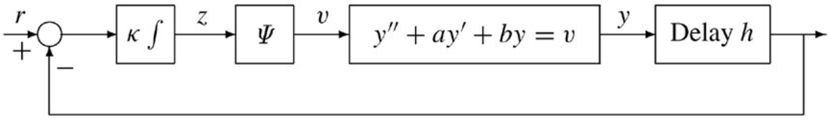

Fig. 1. Integral control of a system with input hysteresis and output delay.

which, on writing $x(\cdot) = z(\cdot + h)$ and introducing the hysteresis operator Φ given by $\Phi(u) = \kappa(\Psi(u) - br)$, can be re-written in the form

$$x'''(t) + ax''(t) + bx'(t) + (\Phi(x(\cdot - h)))(t) = 0.$$

We subsume the above examples in a general investigation of the asymptotic behaviour of systems of each of the following two forms

$$p(D)x + (\Phi(x))(\cdot - h) = 0 \quad \text{or} \quad p(D)x + \Phi(x(\cdot - h)) = 0,$$

where p is a monic polynomial and D is the differential operator. We determine conditions – expressed in terms of p, the delay parameter h and a Lipschitz-type constant associated with the hysteresis operator Φ – which ensure boundedness of solutions and related asymptotic and integrability properties.

We mention that, in the systems and control community, hysteretic effects have received increasing attention in recent years: examples include closed-loop position control of Preisach hysteresis [6], passivity-based control of hysteresis in smart actuators [7], inverse compensation of hysteresis [18,19], micro-positioning control problems using piezo electric actuators [11], H^∞-control of hysteretic systems [17] and integral control in the presence of in input hysteresis [12,14]. Moreover, stability properties of hysteretic systems have been considered by a number of authors in a variety of contexts: for example, sufficient conditions for asymptotic stability of oscillations in nonlinear systems with respect to small hysteretic perturbations have been given in [4], input–output stability results can be found in [2,7,13] and asymptotic/boundedness properties of feedback systems with hysteresis were studied within the framework of absolute stability in [1,2,8,9,13,21]. It is these latter contributions to which the present paper is similar in spirit: in particular, using suitable extensions of results in [13] as a basis, we analyse asymptotic and boundedness properties of the solutions of the differential-delay hysteretic systems formulated above.

The paper is structured as follows. In Section 2, the underlying class of hysteresis operators is defined, some of their known properties are highlighted, some new properties are derived, and several specific examples of commonly-occurring hysteretic nonlinearities are provided. Section 3 assembles suitably generalized results from [13] pertaining to the regularity and asymptotic properties of solutions of a general class of input–output systems (described by a convolution operator) with hysteresis in the feedback loop. The main contributions of the paper reside in Sections 4 and 5. Firstly, in Section 4, the results assembled in Section 3 form the framework for the analysis of differential-delay systems with hysteresis (an existence theory for such systems is developed in Appendix A) wherein conditions are investigated under which a variety of asymptotic properties, and related boundedness and integrability properties, of solutions are guaranteed. Secondly, in Section 5, we return to the specific context of systems of second or third order (akin to the motivating examples above): explicit conditions, in terms of the system parameters, are given under, which the "nice" asymptotic behaviour, alluded to above, is assured.

We conclude this introduction with some remarks on notation.

Notation and terminology. We denote the "punctured" real line by $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $h \geq 0$, define $\mathbb{R}_h := [-h, \infty)$.

Let $c \in \mathbb{R}$ and I be an interval of the form $I = [c, T)$, where $c < T \leq \infty$, or $I = [c, T]$, where $c < T < \infty$. Standard spaces of $\mathbb{R}^n$-valued functions defined on I are written in the form $C(I, \mathbb{R}^n)$, $C^m(I, \mathbb{R}^n)$, $L^p(I, \mathbb{R}^n)$ etc. If $n = 1$, that is, in the case of scalar-valued functions, we simply write $C(I)$, $C^m(I)$, $L^p(I)$ etc. The space of real-valued locally absolutely continuous functions defined on I is denoted by $W^{1,1}_{\text{loc}}(I)$, *i.e.*, a function $f : I \to \mathbb{R}$ is in $W^{1,1}_{\text{loc}}(I)$ if, and only if, there exists a function $g \in L^1_{\text{loc}}(I)$ such that

$$f(t) = f(c) + \int_c^t g(\tau)\mathrm{d}\tau, \quad \forall\, t \in I.$$

Of course, if I is compact, then $L^1_{\text{loc}}(I) = L^1(I)$ and $W^{1,1}_{\text{loc}}(I) = W^{1,1}(I)$. Furthermore, $W^{m,p}(I)$ denotes the space of all real-valued functions f that are $m-1$ times continuously differentiable on I, with locally absolutely continuous derivative of order $m-1$, and such that $f =: f^{(0)}$ together with its derivatives $f^{(k)}$, $k = 1, \ldots, m$, are in $L^p(I)$. Endowed with the norm

$$\|f\|_{W^{m,p}(I)} = \sum_{k=0}^{m} \|f^{(k)}\|_{L^p(I)},$$

$W^{m,p}(I)$ is a Banach space.

Let $\alpha \in \mathbb{R}$. The α-exponentially weighted L^p-space of functions $I \to \mathbb{R}$ is defined as

$$L^p_\alpha(I) := \{f : f(\cdot)\exp(-\alpha\,\cdot) \in L^p(I)\}$$

and is endowed with the norm

$$\|f\|_{L^p_\alpha(I)} = \|f(\cdot)\exp(-\alpha\,\cdot)\|_{L^p(I)} = \left(\int_c^T |\mathrm{e}^{-\alpha t} f(t)|^p \mathrm{d}t\right)^{1/p}.$$

Similarly, we define the α-exponentially weighted $W^{m,p}$-space

$$W^{m,p}_\alpha(I) := \{f : f(\cdot)\exp(-\alpha\,\cdot) \in W^{m,p}_\alpha(I)\},$$

which, endowed with the norm

$$\|f\|_{W^{m,p}_\alpha(I)} = \|f(\cdot)\exp(-\alpha\,\cdot)\|_{W^{m,p}(I)},$$

is a Banach space.

When the correct interpretation is clear from context, we do not distinguish notationally between a function and its restriction: for example, if $u \in C(\mathbb{R}_h)$ and $I \subset \mathbb{R}_h$ is an interval, we write $u \in W^{1,1}_{\text{loc}}(I)$ in place of $u|_I \in W^{1,1}_{\text{loc}}(I)$.

Let $h_1 \geq h_2 \geq 0$. An operator $\Psi : C(\mathbb{R}_{h_1}, \mathbb{R}^n) \to C(\mathbb{R}_{h_2}, \mathbb{R}^n)$ is said to be *causal* if, for all $\tau \geq -h_2$ and all $v_1, v_2 \in C(\mathbb{R}_{h_1}, \mathbb{R}^n)$, $v_1 = v_2$ on $[-h_1, \tau]$ implies that $\Psi(v_1) = \Psi(v_2)$ on $[-h_2, \tau]$. For $h \geq 0$ and $I \subset \mathbb{R}_h$ an interval with $-h \in I$, we define, for each $\tau \in I$, the operator $Q_\tau : C(I, \mathbb{R}^n) \to C(\mathbb{R}_h, \mathbb{R}^n)$ by

$$(Q_\tau u)(t) = \begin{cases} u(t), & -h \leq t \leq \tau, \\ u(\tau), & t > \tau. \end{cases} \tag{1.1}$$

The following simple, but important, remark shows that, for $T \in (-h_2, \infty)$, a causal operator $\Psi : C(\mathbb{R}_{h_1}, \mathbb{R}^n) \to C(\mathbb{R}_{h_2}, \mathbb{R}^n)$ can be "localized" to $C(I, \mathbb{R}^n)$, with $I = [-h_1, T)$ or $I = [-h_1, T]$.

Remark 1.1. Let $h_1 \geq h_2 \geq 0$ and $-h_2 < T < \infty$. If $\Psi : C(\mathbb{R}_{h_1}, \mathbb{R}^n) \to C(\mathbb{R}_{h_2}, \mathbb{R}^n)$ is causal and I_1 (respectively, I_2) is an interval of the form $[-h_1, T)$ or $[-h_1, T]$ (respectively, $[-h_2, T)$ or $[-h_2, T]$), then Ψ "localizes" in a natural way to an operator mapping $C(I_1, \mathbb{R}^n)$ into $C(I_2, \mathbb{R}^n)$: for $v \in C(I_1, \mathbb{R}^n)$, simply set

$$(\Psi(v))(t) := (\Psi(Q_\tau v))(t), \quad -h_2 \leq t \leq \tau, \quad \tau \in I_2.$$

The causality of Ψ guarantees that this definition does not depend on the choice of τ, and so $\Psi(v)$ is a well-defined function in $C(I_2, \mathbb{R}^n)$ for any $v \in C(I_1, \mathbb{R}^n)$. We will not distinguish notationally between the original causal operator and its "localization". $\diamond$

For $I = [-h, T)$, where $-h < T \leq \infty$, or $I = [-h, T]$, where $-h < T < \infty$, define the *delay operator* $\Delta_h : C(I) \to C(I + h)$ by

$$(\Delta_h x)(t) := x(t-h) \quad \forall\, t \in I + h. \tag{1.2}$$

By $\mathcal{B}(X)$, we denote the class of bounded linear operators from a linear normed space X into itself. An operator $G \in \mathcal{B}(L^2(\mathbb{R}_0))$ is said to be *shift-invariant* if

$$S_\tau G = G S_\tau, \quad \forall\, \tau > 0,$$

where $S_\tau \in \mathcal{B}(L^2(\mathbb{R}_0))$ denotes the right shift by τ, that is,

$$(S_\tau u)(t) = \begin{cases} u(t-\tau) & \text{for } t \geq \tau, \\ 0 & \text{for } t < \tau. \end{cases}$$

It is well known (and easy to show) that shift-invariant operators are causal. Moreover, if $G \in \mathcal{B}(L^2(\mathbb{R}_0))$ is shift-invariant, then G has a *transfer function* $\mathbf{G} \in H^\infty$, *i.e.*,

$$\mathcal{L}(Gu) = \mathbf{G}(\mathcal{L}u), \quad \forall\, u \in L^2(\mathbb{R}_0),$$

where H^∞ denotes the algebra of complex-valued bounded analytic functions defined on the open right-half of the complex plane, and $\mathcal{L}$ denotes Laplace transformation. In particular, if G is a convolution operator with kernel $g \in L^1(\mathbb{R}_0)$, then the transfer function of G is equal to the Laplace transform of the kernel, *i.e.*, $\mathbf{G} = \mathcal{L}g$.

Finally, $\theta : \mathbb{R}_h \to \mathbb{R}$ denotes the function which is identically equal to 1; that is, $\theta(t) = 1$ for all $t \in \mathbb{R}_h$.

2. Hysteresis operators

In this section, we define and discuss a class of hysteresis operators. For more information on the mathematical theory of hysteresis operators see, for example, [3,5,10,20]. Our treatment of hysteresis operators has been strongly influenced by [3,5].

2.1. Definitions and properties

Let $h \geq 0$. A function $f : \mathbb{R}_h \to \mathbb{R}_h$ is a *time transformation* (*on* $\mathbb{R}_h$) if f is continuous, non-decreasing with $f(-h) = -h$ and $\lim_{t\to\infty} f(t) = \infty$; in other words, f is a time transformation if it is continuous, non-decreasing and surjective. An operator $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is *rate independent* if, for every time transformation f,

$$\Phi(u \circ f) = \Phi(u) \circ f.$$

Following [3,5,20], we say that $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is a *hysteresis operator* if Φ is causal and rate independent.

The *numerical value set*, NVS Φ, of a hysteresis operator Φ, is defined by

$$\text{NVS}\,\Phi := \{(\Phi(u))(t) : u \in C(\mathbb{R}_h),\ t \in \mathbb{R}_h\}.$$

For $w \in C([-h,\alpha])$ (with $\alpha \geq -h$) and $\gamma, \delta > 0$, we define

$$\mathcal{C}(w; \delta, \gamma) := \left\{ v \in C([-h, \alpha+\gamma]) : v|_{[-h,\alpha]} = w,\ \max_{t\in[\alpha,\alpha+\gamma]} |v(t) - w(\alpha)| \leq \delta \right\}, \tag{2.1}$$

that is, the space of all continuous extensions v of $w \in C([-h,\alpha])$ to the interval $[-h, \alpha+\gamma]$ with the property that $|v(t) - w(\alpha)| \leq \delta$ for all $t \in [\alpha, \alpha+\gamma]$.

A function $u \in C(\mathbb{R}_h)$ is *ultimately non-decreasing (non-increasing)* if there exists $\tau \in \mathbb{R}_h$ such that u is non-decreasing (non-increasing) on $[\tau, \infty)$; u is said to be *approximately ultimately non-decreasing (non-increasing)* if, for all $\varepsilon > 0$, there exists an ultimately non-decreasing (non-increasing) function $v \in C(\mathbb{R}_h)$ such that

$$|u(t) - v(t)| \leq \varepsilon, \quad \forall\, t \in \mathbb{R}_h.$$

We will have occasion to impose some or all of the following six conditions on hysteresis operators $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$:

(N1) If $I \subset \mathbb{R}_h$ is an interval and $u \in C(\mathbb{R}_h) \cap W^{1,1}_{\text{loc}}(I)$, then $\Phi(u) \in W^{1,1}_{\text{loc}}(I)$;

(N2) Φ is monotone in the sense that, if $I \subset \mathbb{R}_h$ is an interval and $u \in C(\mathbb{R}_h) \cap W^{1,1}_{\text{loc}}(I)$, then

$$(\Phi(u))'(t)\, u'(t) \geq 0, \quad \text{a.e. } t \in I;$$

(N3) there exists a $\lambda > 0$ such that, for all $\alpha \geq -h$ and $w \in C([-h,\alpha])$, there exist numbers $\gamma, \delta > 0$ such that

$$\max_{\tau\in[\alpha,\alpha+\gamma]} |(\Phi(u))(\tau) - (\Phi(v))(\tau)| \leq \lambda \max_{\tau\in[\alpha,\alpha+\gamma]} |u(\tau) - v(\tau)|, \quad \forall\, u, v \in \mathcal{C}(w; \delta, \gamma). \tag{2.2}$$

(N4) for all $\alpha \in (-h, \infty)$ and all $u \in C([-h, \alpha))$, there exists $c > 0$ such that

$$\max_{\tau \in [0,t]} |(\Phi(u))(\tau)| \le c(1 + \max_{\tau \in [-h,t]} |u(\tau)|), \quad \forall\, t \in [-h, \alpha).$$

(N5) if $u \in C(\mathbb{R}_h)$ is approximately ultimately non-decreasing and $\lim_{t\to\infty} u(t) = \infty$, then $(\Phi(u))(t)$ and $(\Phi(-u))(t)$ converge, as $t \to \infty$, to $\sup \mathrm{NVS}\, \Phi$ and $\inf \mathrm{NVS}\, \Phi$, respectively;

(N6) if, for $u \in C(\mathbb{R}_h)$, $\lim_{t\to\infty} (\Phi(u))(t) \in \mathrm{int}\, \mathrm{NVS}\, \Phi$, then u is bounded.

Note that, in (N3) and (N4), the functions $\Phi(u)$ and $\Phi(v)$ are well-defined by Remark 1.1. It is not difficult to deduce that (N5) implies that NVS Φ is an interval.

Whilst (some or all of) these technical assumptions are variously invoked in the later analysis, they are natural in the sense that they hold for the most commonly-encountered hysteresis operators: relay, elastic–plastic, backlash, Prandtl, Preisach (see Section 2.2 below for details). Futhermore, we remark that many hysteresis operators (see, for example, [5,12]) are *Lipschitz continuous* in the sense that

$$\sup_{\tau \in \mathbb{R}_h} |(\Phi(u))(\tau) - (\Phi(v))(\tau)| \le \lambda \sup_{\tau \in \mathbb{R}_h} |u(\tau) - v(\tau)|, \quad \forall\, u, v \in C(\mathbb{R}_h), \tag{2.3}$$

for some $\lambda > 0$, in which case (N3) is (trivially) satisfied and, furthermore, (N1) holds (see [12]).

An important and well-known property of hysteresis operators is that they commute with Q_τ for all $\tau \in \mathbb{R}_h$. This implies that, if the input to a hysteresis operator becomes stationary at time τ, then the same is true for the corresponding output. We record the commutativity property, which is an easy consequence of causality and rate independence, as follows.

Proposition 2.1. *If Φ is a hysteresis operator, then*

$$\Phi Q_\tau = Q_\tau \Phi, \quad \forall\, \tau \in \mathbb{R}_h, \tag{2.4}$$

where $Q_\tau : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is given by (1.1).

Proof. Let $\tau \in \mathbb{R}_h$ and $T > \tau$. Define the time transformation $f : \mathbb{R}_h \to \mathbb{R}_h$ by

$$f(t) := \begin{cases} t & \text{on } [-h, \tau] \\ \tau & \text{on } [\tau, T] \\ t - \tau & \text{on } (T, \infty). \end{cases}$$

Then $(Q_\tau v)|_{[-h,T]} = (v \circ f)|_{[-h,T]}$ for all $v \in C(\mathbb{R}_h)$, and so, invoking causality and rate independence, we have

$$(\Phi(Q_\tau u))\,(t) = (\Phi(u \circ f))\,(t) = (\Phi(u) \circ f)\,(t) = (Q_\tau \Phi(u))\,(t) \quad \forall\, t \in [-h, T], \forall\, u \in C(\mathbb{R}_h).$$

Since $\tau \in \mathbb{R}_h$ and $T > \tau$ are arbitrary, (2.4) follows. □

Section 4 (the main contribution) of the paper is devoted to a study of differential systems with delay impinging on the hysteresis in one of the forms $\Delta_h \Phi(u)$ or $\Phi(\Delta_h u)$. Anticipating the methodology of Section 4, which involves the reduction of systems with the latter type of delay to systems with the former type of delay, we proceed to associate, with any hysteresis operator Φ (with domain $C(\mathbb{R}_0)$), a hysteresis operator $\tilde{\Phi}$ (with domain $C(\mathbb{R}_h)$): the operator $\tilde{\Phi}$ plays a central role in the reduction.

Let $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ be a hysteresis operator, and let $h \ge 0$. Define $\tilde{\Phi} : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ by

$$(\tilde{\Phi}(u))(t) := (\Phi(\Delta_h u))(t + h), \quad \forall\, t \ge -h, \forall\, u \in C(\mathbb{R}_h) \tag{2.5}$$

and note that

$$(\Phi(\Delta_h u))(t) = (\Delta_h \tilde{\Phi}(u))(t), \quad \forall\, t \ge 0, \forall\, u \in C(\mathbb{R}_h). \tag{2.6}$$

Proposition 2.2. *Let $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ be a hysteresis operator, let $h \ge 0$ and let $\tilde{\Phi} : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ be the associated operator defined by (2.5). Then $\tilde{\Phi}$ is a hysteresis operator. Moreover, if Φ satisfies some or all of the hypotheses* (N1)−(N6) *(in the context of operators with domain $C(\mathbb{R}_0)$), then $\tilde{\Phi}$ satisfies the same hypotheses (in the context of operators with domain $C(\mathbb{R}_h)$).*

Proof. The proof of causality of $\tilde{\Phi}$ is trivial and is therefore omitted. We will prove rate-independence. Let f_h be a time transformation on $\mathbb{R}_h$. Define a time transformation on $\mathbb{R}_0$ by

$$f(t) := f_h(t-h)+h, \quad \forall\, t \geq 0.$$

Let $u \in C(\mathbb{R}_h)$. Then,

$$((\Delta_h u)\circ f)(t) = (\Delta_h u)(f(t)) = u(f(t)-h) = u(f_h(t-h)) = (u\circ f_h)(t-h), \quad \forall\, t \geq 0,$$

whence

$$(\Delta_h u)\circ f = \Delta_h(u\circ f_h). \tag{2.7}$$

By (2.5), together with rate-independence of Φ and (2.7), we have

$$\begin{aligned}(\tilde{\Phi}(u))(f_h(t)) &= (\Phi(\Delta_h u))(f_h(t)+h) = (\Phi(\Delta_h u))(f(t+h)) = (\Phi((\Delta_h u)\circ f))(t+h)\\ &= (\Phi(\Delta_h(u\circ f_h)))(t+h) = (\tilde{\Phi}(u\circ f_h))(t), \quad \forall\, t \geq -h,\end{aligned}$$

proving the rate-independence of $\tilde{\Phi}$. Finally, it is straightforward to show that if Φ satisfies some or all of the hypotheses (N1)–(N6), then $\tilde{\Phi}$ satisfies the same hypotheses. □

Let $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ be a hysteresis operator. For every $x \in \mathbb{R}$, set

$$R_\Phi(x) := \{(\Phi(u))(1) : u \in C(\mathbb{R}_h),\, u(1) = x\}.$$

Define $\varphi_- : \mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ and $\varphi_+ : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ by

$$\varphi_-(x) := \inf R_\Phi(x) \quad \text{and} \quad \varphi_+(x) := \sup R_\Phi(x),$$

respectively. It follows from rate-independence that

$$\{(\Phi(u))(t) : u \in C(\mathbb{R}_h),\, u(t) = x\} = R_\Phi(x), \quad \forall\, x \in \mathbb{R}, \forall t \in (-h,\infty). \tag{2.8}$$

Moreover, using causality and Proposition 2.1, we see that

$$(\Phi(u))(-h) = (\Phi(u(-h)\theta))(-h) = (\Phi(u(-h)\theta))(1)$$

and thus,

$$\{(\Phi(u))(-h) : u \in C(\mathbb{R}_h),\, u(-h) = x\} \subset R_\Phi(x), \quad \forall\, x \in \mathbb{R}.$$

Consequently,

$$\varphi_-(u(t)) \leq (\Phi(u))(t) \leq \varphi_+(u(t)), \quad \forall\, t \in \mathbb{R}_h, \forall\, u \in C(\mathbb{R}_h).$$

We say that the pair (φ_-, φ_+) is the *envelope* of Φ. It follows from (2.8) that the envelope is "tight". For $\xi \in \mathbb{R}$, we define

$$L_\Phi(\xi) := \{x \in \mathbb{R} : \varphi_-(x) \leq \xi \leq \varphi_+(x)\}. \tag{2.9}$$

Proposition 2.3. *Assume that Φ is a Lipschitz continous hysteresis operator (that is, (2.3) is satisfied for some $\lambda > 0$) with envelope (φ_-, φ_+). Then the following statements hold:*

(i) *If $x \in \mathbb{R}$ is such that $\varphi_-(x) > -\infty$, then φ_- is continuous at x.*
(ii) *If $x \in \mathbb{R}$ is such that $\varphi_+(x) < \infty$, then φ_+ is continuous at x.*
(iii) *If $u \in C(\mathbb{R}_h)$ is bounded and $\lim_{t\to\infty}(\Phi(u))(t) = \xi$ for some $\xi \in \mathbb{R}$, then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), L_\Phi(\xi)) \to 0.$$

Proof. We prove assertion (i) by a contradiction argument. Suppose that the claim is not true. Then there exist $x \in \mathbb{R}$, a sequence (x_n) in $\mathbb{R}$ and $\varepsilon > 0$ such that $\lim_{n\to\infty} x_n = x$ and

$$\text{(a) } \varphi_-(x_n) - \varphi_-(x) \leq -\varepsilon, \quad \forall n \in \mathbb{N} \quad \text{or} \quad \text{(b) } \varphi_-(x_n) - \varphi_-(x) \geq \varepsilon, \quad \forall n \in \mathbb{N}.$$

If (a) holds, then there exists a sequence (v_n) in $C(\mathbb{R}_h)$ with $v_n(1) = x_n$ for all $n \in \mathbb{N}$ and such that

$$(\Phi(v_n))(1) - \varphi_-(x) \le -\varepsilon/2, \quad \forall\, n \in \mathbb{N}.$$

Since $x_n \to x$ as $n \to \infty$, $u_n := v_n - x_n + x \in C(\mathbb{R}_h)$ is such that $u_n(1) = x$ for all $n \in \mathbb{N}$ and $\sup_{t\in\mathbb{R}_h} |v_n(t) - u_n(t)| \to 0$ as $n \to \infty$. Using the Lipschitz continuity of Φ, we obtain that $(\Phi(v_n))(1) - (\Phi(u_n))(1) \to 0$ as $n \to \infty$. Consequently, there exists $N \in \mathbb{N}$ such that

$$(\Phi(u_n))(1) \le \varphi_-(x) - \varepsilon/4, \quad \forall\, n \ge N,$$

which yields the desired contradiction, since, by the definition of φ_-, $(\Phi(u_n))(1) \ge \varphi_-(x)$ for all $n \in \mathbb{N}$.

If (b) holds, then there exists a sequence (w_n) in $C(\mathbb{R}_h)$ with $w_n(1) = x$ for all $n \in \mathbb{N}$ and such that

$$(\Phi(w_n))(1) - \varphi_-(x_n) \le -\varepsilon/2, \quad \forall\, n \in \mathbb{N}.$$

Since $x_n \to x$ as $n \to \infty$, $u_n := w_n - x + x_n \in C(\mathbb{R}_h)$ such that $u_n(1) = x_n$ for all $n \in \mathbb{N}$ and $\sup_{t\in\mathbb{R}_h} |w_n(t) - u_n(t)| \to 0$ as $n \to \infty$. By the Lipschitz continuity of Φ, we have $(\Phi(w_n))(1) - (\Phi(u_n))(1) \to 0$ as $n \to \infty$. Consequently, there exists $N \in \mathbb{N}$ such that

$$(\Phi(u_n))(1) \le \varphi_-(x_n) - \varepsilon/4, \quad \forall\, n \ge N,$$

which yields the desired contradiction, since, by the definition of φ_-, $(\Phi(u_n))(1) \ge \varphi_-(x_n)$ for all $n \in \mathbb{N}$.

Assertion (ii) follows by an argument analogous to that used in establishing assertion (i).

It remains to prove assertion (iii). Again seeking a contradiction, suppose that the claim is not true. Since u is bounded, we conclude that there exist a sequence (t_n) in $\mathbb{R}_h$ and $u^* \in \mathbb{R}$ such that $t_n \to \infty$ and $u(t_n) \to u^*$ as $n \to \infty$, and $\varphi_-(u^*) > \xi$ or $\varphi_+(u^*) < \xi$. Invoking assertions (i) and (ii), it follows that $\liminf_{n\to\infty} |(\Phi(u))(t_n) - \xi| > 0$, contradicting the fact that $(\Phi(u))(t)$ converges to ξ as $t \to \infty$. □

2.2. *Examples of hysteresis operators*

Below, we describe some important classes of hysteresis operators satisfying (N1)–(N6).

Relay (passive, positive) hysteresis. *Relay* (also called *passive* or *positive*) hysteresis, is discussed in a mathematically rigorous context in a number of references; see for example [12] and [16]. Fig. 2 illustrates schematically the action of relay hysteresis, one of the simplest of hysteretic phenomena. The corresponding operator is denoted by $\mathcal{R}_\zeta$. Whilst conceptually clear, a precise mathematical definition of relay hysteresis is notationally cumbersome. Let $b_0, b_1 \in \mathbb{R}$ with $b_0 < b_1$, and let $\varphi_0 : [b_0, \infty) \to \mathbb{R}$ and $\varphi_1 : (-\infty, b_1] \to \mathbb{R}$ be non-decreasing and globally Lipschitz with the same Lipschitz constant $l \ge 0$ and such that $\varphi_0(b_0) = \varphi_1(b_0)$ and $\varphi_0(b_1) = \varphi_1(b_1)$. For $u \in C(\mathbb{R}_h)$ and $t \ge -h$, define

$$S(u,t) := u^{-1}(\{b_0, b_1\}) \cap [-h, t], \quad \tau(u,t) := \begin{cases} \max S(u,t) & \text{if } S(u,t) \ne \emptyset, \\ -1 & \text{if } S(u,t) = \emptyset. \end{cases}$$

Following [16], for $\zeta \in \{0, 1\}$, we define an operator $\mathcal{R}_\zeta : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ by

$$(\mathcal{R}_\zeta(u))(t) = \begin{cases} \varphi_1(u(t)) & \text{if } u(t) \le b_0, \\ \varphi_0(u(t)) & \text{if } u(t) \ge b_1, \\ \varphi_1(u(t)) & \text{if } u(t) \in (b_0, b_1),\ \tau(u,t) \ne -1 \text{ and } u(\tau(u,t)) = b_0, \\ \varphi_0(u(t)) & \text{if } u(t) \in (b_0, b_1),\ \tau(u,t) \ne -1 \text{ and } u(\tau(u,t)) = b_1, \\ \varphi_0(u(t)) & \text{if } u(t) \in (b_0, b_1),\ \tau(u,t) = -1 \text{ and } \zeta = 0, \\ \varphi_1(u(t)) & \text{if } u(t) \in (b_0, b_1),\ \tau(u,t) = -1 \text{ and } \zeta = 1. \end{cases} \tag{2.10}$$

The number ζ plays the role of an "initial state", which determines the output value $(\mathcal{R}_\zeta(u))(t)$ if $u(s) \in (b_0, b_1)$ for all $s \in [-h, t]$. From [12], we know that $\mathcal{R}_\zeta$ satisfies (N1)–(N6) (with $\lambda = l$ in (N3)). We note that $\mathrm{NVS}\,\mathcal{R}_\zeta = \mathrm{im}\,\varphi_0 \cup \mathrm{im}\,\varphi_1$. It is clear that $\mathcal{R}_\zeta$ is Lipschitz continuous if and only if $\varphi_0(v) = \varphi_1(v)$ for all $v \in [b_0, b_1]$, *i.e.*, if and only if $\mathcal{R}_\zeta$ "degenerates" into a static nonlinearity.

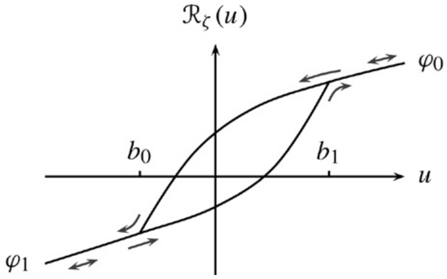

Fig. 2. Relay hysteresis.

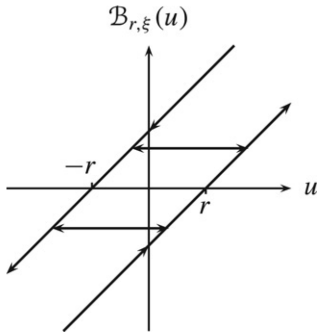

Fig. 3. Backlash hysteresis.

Backlash hysteresis (play operator). A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references, see for example [3,5,10] and [12]. Let $r \in \mathbb{R}_0$ and introduce the function

$$b_r : \mathbb{R}^2 \to \mathbb{R}, \quad (v, w) \mapsto \max\{v - r,\ \min\{v + r, w\}\} = \begin{cases} v - r, & \text{if } w < v - r \\ w, & \text{if } w \in [v - r, v + r] \\ v + r, & \text{if } w > v + r. \end{cases}$$

Let $C_{\text{pm}}(\mathbb{R}_h)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_h$. For all $r \in \mathbb{R}_0$ and all $\xi \in \mathbb{R}$, we define the operator $\mathcal{B}_{r,\xi} : C_{\text{pm}}(\mathbb{R}_h) \to C(\mathbb{R}_h)$ by

$$(\mathcal{B}_{r,\xi}(u))(t) = \begin{cases} b_r(u(0), \xi) & \text{for } t = -h, \\ b_r(u(t), (\mathcal{B}_{r,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \ldots, \end{cases}$$

where $-h = t_0 < t_1 < t_2 < \ldots$, $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ξ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition (t_i). Fig. 3 illustrates how $\mathcal{B}_{r,\xi}$ acts. It is well known that $\mathcal{B}_{r,\xi}$ extends to a Lipschitz continuous operator on $C(\mathbb{R}_h)$ (with Lipschitz constant $\lambda = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{r,\xi}$. It is also well known (and easy to check) that $\mathcal{B}_{r,\xi}$ is a hysteresis operator.

As shown, for example, in [12], $\mathcal{B}_{r,\xi}$ satisfies (N1)–(N6) (with $\lambda = 1$ in (N3)). It is obvious that NVS $\mathcal{B}_{r,\xi} = \mathbb{R}$.

Elastic–plastic hysteresis (stop operator). The *elastic–plastic* operator (also called *stop* operator) is discussed in a mathematically rigorous context in a number of references; see for example [3,5,10] and [12]. To give a formal definition of the elastic–plastic operator, for each $r \in \mathbb{R}_0$, define the function $e_r : \mathbb{R} \to \mathbb{R}$ by

$$e_r(v) = \min\{r,\ \max\{-r, v\}\} = \begin{cases} -r, & \text{if } v < -r \\ v, & \text{if } v \in [-r, r] \\ r, & \text{if } v > r. \end{cases}$$

For all $r \in \mathbb{R}_0$ and all $\xi \in \mathbb{R}$, we define an operator $\mathcal{E}_{r,\xi} : C_{\text{pm}}(\mathbb{R}_h) \to C(\mathbb{R}_h)$ by

$$(\mathcal{E}_{r,\xi}(u))(t) = \begin{cases} e_r(u(0) - \xi) & \text{for } t = -h, \\ e_r(u(t) - u(t_i) + (\mathcal{E}_{r,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \ldots, \end{cases}$$

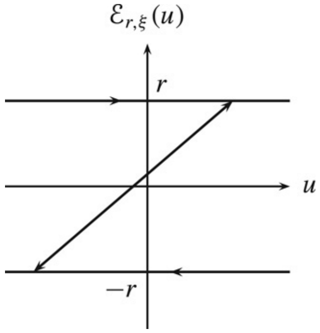

Fig. 4. Elastic–plastic hysteresis.

where $-h = t_0 < t_1 < t_2 < \ldots$, $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval $[t_i, t_{i+1}]$. Again, ξ plays the role of an "initial state". The operators $\mathcal{E}_{r,\xi}$ and $\mathcal{B}_{r,\xi}$ are closely related:

$$\mathcal{E}_{r,\xi}(u) + \mathcal{B}_{r,\xi}(u) = u, \quad \forall\, u \in C_{\mathrm{pm}}(\mathbb{R}_h), \tag{2.11}$$

see, for example, [5, p. 44]. Fig. 4 illustrates how $\mathcal{E}_{r,\xi}$ acts. It follows from (2.11) and the properties of $\mathcal{B}_{r,\xi}$ that $\mathcal{E}_{r,\xi}$ extends to a Lipschitz continuous hysteresis operator on $C(\mathbb{R}_h)$ (with Lipschitz constant $\lambda = 2$). This extension, which we denote by the same symbol $\mathcal{E}_{r,\xi}$, is called the elastic–plastic operator.

As shown, for example, in [12], $\mathcal{E}_{r,\xi}$ satisfies (N1)–(N6) (with $\lambda = 2$ in (N3)). Trivially, NVS $\mathcal{E}_{r,\xi} = [-r, r]$.

Prandtl and Preisach operators. All the hysteresis operators considered so far model relatively simple hysteresis loops. The Preisach operator described below, encompasses both backlash, elastic–plastic and, more generally, Prandtl operators. It represents a far more general type of hysteresis which, for certain input functions, exhibits nested loops in the corresponding input–output characteristics. Let $\xi : \mathbb{R}_0 \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Furthermore, let μ be a signed Borel measure on $\mathbb{R}_0$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_0$, where $|\mu|$ denotes the total variation of μ. Denoting the Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function, and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ defined by

$$(\mathcal{P}_\xi(u))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{r,\xi(r)}(u))(t)} w(s,r)\mu_L(\mathrm{d}s)\mu(\mathrm{d}r) + w_0, \quad \forall\, u \in C(\mathbb{R}_h),\ \forall t \in \mathbb{R}_h, \tag{2.12}$$

is called a *Preisach* operator, cf. [5, p. 55]. It is clear that $\mathcal{P}_\xi$ is a hysteresis operator. Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $|\mu|(\mathbb{R}_0)\|w\|_\infty$; see [12]. Furthermore, if we additionally assume that μ and w are non-negative, then, as shown in [12], (N1)–(N6) hold (with $\lambda = \mu(\mathbb{R}_0)\|w\|_\infty$ in (N3)).

Setting $w(s,r) \equiv 1$ and $w_0 = 0$ in (2.12), we obtain the *Prandtl* operator

$$(\mathcal{P}_\xi(u))(t) = \int_0^\infty (\mathcal{B}_{r,\xi(r)}(u))(t)\mu(\mathrm{d}r), \quad \forall\, u \in C(\mathbb{R}_h), \forall t \in \mathbb{R}_h. \tag{2.13}$$

For $h = 0$, $\xi \equiv 0$ and the measure μ given by $\mu(E) = \mu_L(E \cap [0,5])$, the Prandtl operator (2.13) becomes

$$(\mathcal{P}_0(u))(t) = \int_0^5 (\mathcal{B}_{r,0}(u))(t)\mathrm{d}r, \quad \forall\, u \in C(\mathbb{R}_0), \forall t \in \mathbb{R}_0$$

and is illustrated in Fig. 5.

Finally, we mention that Preisach operators can often be written as "weighted sums" of relay switches. More precisely, if μ is the Lebesgue measure on $\mathbb{R}_0$, $w \in L^1(\mathbb{R} \times \mathbb{R}_0)$, $\xi = 0$ and

$$w_0 = \frac{1}{2}\left(\int_0^\infty \int_{-\infty}^0 w(s,r)\mathrm{d}s\mathrm{d}r - \int_0^\infty \int_0^\infty w(s,r)\mathrm{d}s\mathrm{d}r\right),$$

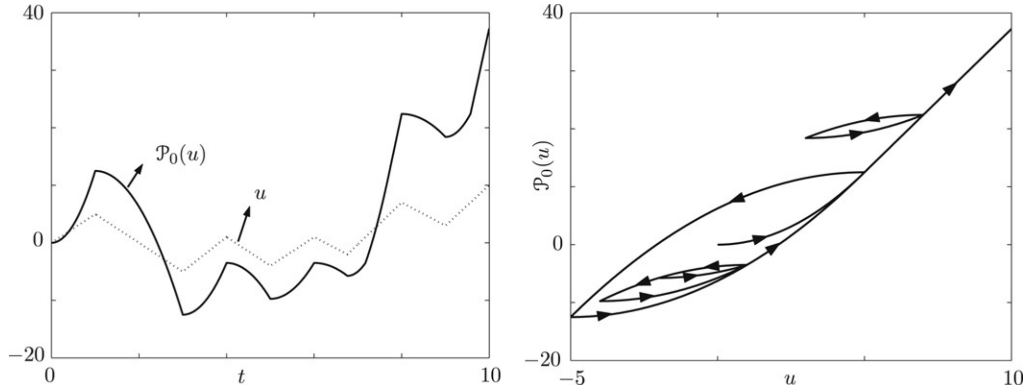

Fig. 5. Example of Prandtl hysteresis.

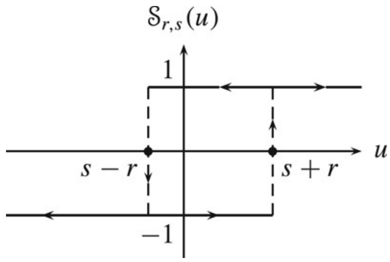

Fig. 6. Relay switch hysteresis.

then the Preisach operator (2.12) can be written as

$$u(\cdot) \mapsto \frac{1}{2}\left(\int_0^\infty \int_{-\infty}^\infty w(s,r)\left(\mathcal{S}_{r,s}(u)\right)(\cdot)\mathrm{d}s\mathrm{d}r\right), \quad \forall u \in C(\mathbb{R}_h),$$

where $\mathcal{S}_{r,s}$ is the *relay switch* operator illustrated in Fig. 6 (see [5] for details). More formally, $\left(\mathcal{S}_{r,s}(u)\right)(t)$ is given by the right-hand side of (2.10) with $b_0 = s - r$, $b_1 = s + r$, $\varphi_0(v) \equiv 1$, $\varphi_1(v) \equiv -1$ and

$$\zeta = \zeta(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$$

2.3. *Generalized sector condition*

Definition 2.4. An operator $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is said to satisfy a *generalized sector condition* if there exist $\tau_1 \le 0$, $\tau_2 \ge 0$ and $\sigma \in \mathbb{R}$ such that, for all $u \in C(\mathbb{R}_h)$ and $t \ge -h$,

$$u(t) \ge \tau_2 \Rightarrow (\Phi(u))(t) \ge \sigma u(t),$$
$$u(t) \le \tau_1 \Rightarrow (\Phi(u))(t) \le \sigma u(t).$$

If $\sigma > 0$, then we say that Φ satisfies a *positive generalized sector condition*. ◇

For every $\xi \in \mathbb{R}$ and $r > 0$, the backlash operator $\mathcal{B}_{r,\xi}$ satisfies a positive generalized sector condition with arbitrary $0 < \sigma < 1$ and $\tau_1 = -r/(1-\sigma)$, $\tau_2 = r/(1-\sigma)$ and the elastic–plastic operator satisfies a generalized sector condition with arbitrary $-\infty < \sigma < 0$ and $\tau_1 = r/\sigma$, $\tau_2 = -r/\sigma$.

We show that, under certain assumptions, the Preisach operator $\mathcal{P}_\xi$ defined in (2.12) (and hence, *a fortiori*, the Prandlt operator defined in (2.13)) satisfies a positive generalized sector condition.

Proposition 2.5. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in (2.12). Assume that the measure μ is non-negative, with*

$$0 < \rho := \mu(\mathbb{R}_0) < \infty \quad \textit{and} \quad 0 < \alpha := \int_0^\infty r\mu(\mathrm{d}r) < \infty.$$

Assume further that

$$0 < \beta_1 := \text{ess inf}_{(s,r)\in\mathbb{R}\times\mathbb{R}_0} w(s,r) \quad \textit{and} \quad \beta_2 := \text{ess sup}_{(s,r)\in\mathbb{R}\times\mathbb{R}_0} w(s,r) < \infty.$$

Then, for every $0 < \varepsilon < \rho\beta_1$, *we have*

$$|u(t)| \geq (\alpha\beta_2 + |w_0|)/\varepsilon \Longrightarrow (\mathcal{P}_\xi(u))(t)u(t) \geq (\rho\beta_1 - \varepsilon)u^2(t), \quad \forall u \in C(\mathbb{R}_h), \forall t \in \mathbb{R}_h, \tag{2.14}$$

i.e., $\mathcal{P}_\xi$ *satisfies a positive generalized sector condition.*

Proof. Note initially that, by the definition of the backlash operator, we have

$$(\mathcal{B}_{r,\xi(r)}(u))(t) \in [u(t) - r, u(t) + r] \quad \forall u \in C(\mathbb{R}_h),\ \forall t \in \mathbb{R}_h, \forall r \in \mathbb{R}_0.$$

Let $\varepsilon \in (0, \rho\beta_1)$ and define $\tau := (\alpha\beta_2 + |w_0|)/\varepsilon$. Let $u \in C(\mathbb{R}_h)$ and $t \in \mathbb{R}_h$.
Case 1. Assume that $u(t) \geq \tau$. Writing $E_1 = [0, u(t)]$ and $E_2 = (u(t), \infty)$, it follows that

$$\begin{aligned}(\mathcal{P}_\xi u)(t) &\geq \left(\int_{E_1} + \int_{E_2}\right)\int_0^{u(t)-r} w(s,r)\mu_L(\mathrm{d}s)\mu(\mathrm{d}r) - |w_0| \\ &\geq \beta_1 \int_{E_1} (u(t) - r)\,\mu(\mathrm{d}r) + \beta_2 \int_{E_2} (u(t) - r)\,\mu(\mathrm{d}r) - |w_0| \\ &= (\beta_1\mu(E_1) + \beta_2\mu(E_2))\,u(t) - \beta_1 \int_{E_1} r\,\mu(\mathrm{d}r) - \beta_2 \int_{E_2} r\,\mu(\mathrm{d}r) - |w_0| \\ &\geq \rho\beta_1 u(t) - \alpha\beta_2 - |w_0| = \rho\beta_1 u(t) - \varepsilon\tau,\end{aligned}$$

and so we may conclude that

$$u(t) \geq \tau \Longrightarrow (\mathcal{P}_\xi u)(t) \geq (\rho\beta_1 - \varepsilon)u(t). \tag{2.15}$$

Case 2. Now assume that $u(t) \leq -\tau$. Writing $E_1 = [0, -u(t)]$ and $E_2 = (-u(t), \infty)$, we have

$$\begin{aligned}(\mathcal{P}_\xi u)(t) &\leq \left(\int_{E_1} + \int_{E_2}\right)\int_0^{u(t)+r} w(s,r)\mu_L(\mathrm{d}s)\mu(\mathrm{d}r) + |w_0| \\ &\leq \beta_1 \int_{E_1} (u(t) + r)\,\mu(\mathrm{d}r) + \beta_2 \int_{E_2} (u(t) + r)\,\mu(\mathrm{d}r) + |w_0| \leq \rho\beta_1 u(t) + \varepsilon\tau,\end{aligned}$$

from which we may infer that

$$u(t) \leq -\tau \Longrightarrow (\mathcal{P}_\xi u)(t) \leq (\rho\beta_1 - \varepsilon)u(t). \tag{2.16}$$

Since $u \in C(\mathbb{R}_h)$ and $t \in \mathbb{R}_h$ are arbitrary, the conjunction of (2.15) and (2.16) yields (2.14). □

Example 2.6. The Prandtl operator shown in Fig. 5 satisfies a positive generalized sector condition: in particular, (2.14) holds with $\rho = 5$, $\alpha = 25/2$, $\beta_1 = 1 = \beta_2$ and $\varepsilon \in (0, 5)$. ◇

Remark 2.7. If the (signed Borel) measure μ satisfies any of the following three conditions

(i) μ is non-negative, regular and has compact support,
(ii) $\mu = \sum_{n=0}^{\infty} \gamma_n\delta_{r_n}$, where δ_{r_n} is the unit mass at $r_n \geq 0$, $\gamma_n > 0$, and $\sum_{n=0}^{\infty} \gamma_n r_n < \infty$,
(iii) $\mu(\mathrm{d}r) = f(r)\mathrm{d}r$, with $f \geq 0$, $f \in L^1(\mathbb{R}_0)$ and $f(r) = O(r^{-(2+\varepsilon)})$ as $r \to \infty$ for some $\varepsilon > 0$,

then the hypotheses on μ in Proposition 2.5 hold. ◇

3. Asymptotic behaviour of solutions of feedback systems with hysteresis

In the following, we derive generalizations of certain results from [13], the purpose of which is to facilitate the main investigation in Section 4.

We first consider the feedback system shown in Fig. 7, where G is a convolution operator with kernel g, $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is a hysteresis operator, r_1 and r_2 are input and disturbance signals, respectively.

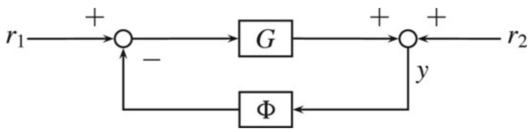

Fig. 7. Feedback system with hysteretic nonlinearity.

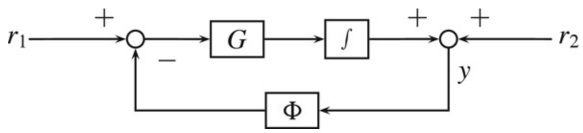

Fig. 8. Feedback system with integrator and hysteresis.

This system is described by

$$y = g * r_1 + r_2 - g * \Phi(y), \quad y|_{[-h,0]} = \varphi \in C([-h, 0]), \quad r_2(0) = \varphi(0), \tag{3.1}$$

where $*$ denotes convolution. A *solution* of (3.1) on $[-h, T)$ for some $0 < T \le \infty$ is a function $y \in C([-h, T))$ satisfying (3.1).

The following result extends part (a) of Theorem 4.1 in [13], which pertains to the case $h = 0$, to the general case of $h \ge 0$: the essence is to show that, by a suitable re-formulation of the problem, the latter case can be reduced to the former.

Theorem 3.1. *Let $g \in L^2_\alpha(\mathbb{R}_0)$ for some $\alpha < 0$ be a function of locally bounded variation. Let $r_1, r_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0)$ with $r_1', r_2' \in L^2_\alpha(\mathbb{R}_0)$ and $r_2(0) = \varphi(0)$. Let Φ satisfy* (N1), (N2) *and* (N3) *with associated constant $\lambda > 0$. Assume that*

$$\inf_{\omega \in \mathbb{R}} \operatorname{Re} \mathbf{G}(\mathrm{i}\omega) > -1/\lambda, \tag{3.2}$$

where $\mathbf{G}$ denotes the Laplace transform of g (or, equivalently, $\mathbf{G}$ is the transfer function of G). Then (3.1) has a unique solution on $\mathbb{R}_h$ which is locally absolutely continuous on $\mathbb{R}_0$, and there exist constants $\beta \in (\alpha, 0)$ and $\gamma > 0$ (both dependent on g and λ, but independent of r_1, r_2 and φ) such that

$$\begin{aligned} &\|y\|_{L^\infty(\mathbb{R}_0)} + \|\Phi(y)\|_{L^\infty(\mathbb{R}_0)} + \|y'\|_{L^2_\beta(\mathbb{R}_0)} + \|(\Phi(y))'\|_{L^2_\beta(\mathbb{R}_0)} \\ &\quad \le \gamma (\|r_1'\|_{L^2_\beta(\mathbb{R}_0)} + \|r_2'\|_{L^2_\beta(\mathbb{R}_0)} + |r_1(0)| + |r_2(0)| + |(\Phi(\varphi))(0)|) \end{aligned} \tag{3.3}$$

and $y(t)$, $(\Phi(y))(t)$ converge, at exponential rates, to finite limits as $t \to \infty$.

In Theorem 3.1, it is assumed that the linear component of the system is described by a convolution operator with kernel in $L^2_\alpha(\mathbb{R}_0)$ for some $\alpha < 0$. This implies that the linear subsystem is input–output stable and, in particular, does not contain any integrators.

The next result, Theorem 3.2, applies to a class of linear systems containing an integrator. Before stating Theorem 3.2, we consider the feedback system shown on Fig. 8, where the operator $G \in \mathcal{B}(L^2(\mathbb{R}_0))$ is assumed to be shift-invariant and $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is a hysteresis operator. The integral equation

$$y(t) = \int_0^t (Gr_1)(\tau)\mathrm{d}\tau + r_2(t) - \int_0^t (G\Phi(y))(\tau)\mathrm{d}\tau, \qquad y|_{[-h,0]} = \varphi \in C([-h, 0]), \quad r_2(0) = \varphi(0) \tag{3.4}$$

describes the system shown in Fig. 8. The solution concept for the initial-value problem (3.4) is the same as that for (3.1). We impose the following assumption on G:

(L) The limit $\mathbf{G}(0) := \lim_{s \to 0, \operatorname{Re} s > 0} \mathbf{G}(s)$ exists, $\mathbf{G}(0) > 0$ and

$$\limsup_{s \to 0, \operatorname{Re} s > 0} |(\mathbf{G}(s) - \mathbf{G}(0))/s| < \infty.$$

The following result extends Theorem 4.5 of [13], which pertains to the case $h = 0$, to the general case of $h \ge 0$.

Theorem 3.2. *Assume that* **G** *(the transfer function of G) satisfies assumption* (L), Φ *satisfies* (N1)−(N5) *and* $0 \in \operatorname{clos}\operatorname{NVS}\Phi$. *Let* $\lambda > 0$ *be the constant associated with* (N3). *Assume that*

$$\inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\frac{\mathbf{G}(\mathrm{i}\omega)}{\mathrm{i}\omega} > -\frac{1}{\lambda}. \tag{3.5}$$

Then, for all $r_1 \in L^2(\mathbb{R}_0)$ *and* $r_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0)$ *with* $r_2' \in L^2(\mathbb{R}_0)$, $r_2(0) = \varphi(0)$, *there exists a unique solution of* (3.4) *defined on* $\mathbb{R}_h$ *which is locally absolutely continuous on* $\mathbb{R}_0$. *Furthermore,* $\lim_{t\to\infty}(\Phi(y))(t) = 0$, *and there exists a constant* $\gamma > 0$ *(depending only on G and* λ*, but not on* r_1, r_2 *and* φ*) such that*

$$\|\Phi(y)\|_{L^\infty(\mathbb{R}_0)} + \|(\Phi(y))'\|_{L^2(\mathbb{R}_0)} \le \gamma(\|r_1\|_{L^2(\mathbb{R}_0)} + \|r_2'\|_{L^2(\mathbb{R}_0)} + |(\Phi(\varphi))(0)|). \tag{3.6}$$

Under the additional assumptions that (N6) *holds and* $0 \in \operatorname{int}\operatorname{NVS}\Phi$, y *is bounded.*

Proof of Theorems 3.1 and 3.2. For the particular case of $h = 0$, proofs of Theorems 3.1 and 3.2 can be found in [13]: specifically, Theorem 3.1 is subsumed by part (a) of Theorem 4.1 in [13], and Theorem 3.2 is subsumed by Theorem 4.5 in [13]. We complete the proof by showing that the general case of $h \ge 0$ can be reduced to the particular case of $h = 0$. To this end, with each $u \in C(\mathbb{R}_0)$, we associate a function $\tilde{u} \in C(\mathbb{R}_h)$ defined by

$$\tilde{u}(t) = \begin{cases} \varphi(t) - \varphi(0) + u(0), & t \in [-h, 0) \\ u(t), & t \ge 0. \end{cases}$$

Now define an operator $\Psi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ by

$$(\Psi(u))(t) := (\Phi(\tilde{u}))(t), \quad \forall\, u \in C(\mathbb{R}_0), \forall t \in \mathbb{R}_0.$$

It is routine to show that Ψ is a hysteresis operator; moreover, if Φ satisfies some or all of the hypotheses (N1)–(N6) (in the context of operators with domain $C(\mathbb{R}_h)$), then Ψ satisfies the same hypotheses (in the context of operators with domain $C(\mathbb{R}_0)$); furthermore, if Φ satisfies (N5), then the interiors of the numerical value sets of Ψ and Φ coincide. It is clear that a function $y \in C(\mathbb{R}_h)$ solves (3.1) if, and only if, the restriction $y_0 = y|_{\mathbb{R}_0}$ satisfies

$$y_0 = g * r_1 + r_2 - g * \Psi(y_0). \tag{3.7}$$

Consequently, an application of part (a) of Theorem 4.1 from [13] to (3.7) establishes the assertions of Theorem 3.1. Similarly, $y \in C(\mathbb{R}_h)$ solves (3.4) if, and only if, the restriction $y_0 = y|_{\mathbb{R}_0}$ satisfies

$$y_0(t) = \int_0^t (Gr_1)(\tau)\mathrm{d}\tau + r_2(t) - \int_0^t (G\,\Psi(y_0))(\tau)\mathrm{d}\tau, \quad \forall\, t \in \mathbb{R}_0. \tag{3.8}$$

Theorem 4.5 in [13], applied to (3.8), yields the assertions of Theorem 3.2. □

4. Differential-delay systems with hysteresis

We now arrive at the main focus of the paper. In particular, we will investigate boundedness and related asymptotic and integrability properties of solutions of the following two initial-value problems:

$$p(D)x + \Delta_h \Phi(x) = 0, \tag{4.1a}$$

$$x|_{[-h,0]} = \varphi \in C([-h, 0]), \quad x^{(k)}(0) = x_k,\ k = 1, \ldots, n-1 \tag{4.1b}$$

and

$$p(D)x + \Phi(\Delta_h x) = 0, \tag{4.2a}$$

$$x|_{[-h,0]} = \varphi \in C([-h, 0]), \quad x^{(k)}(0) = x_k,\ k = 1, \ldots, n-1. \tag{4.2b}$$

Here $h \ge 0$, $p(s) := \sum_{k=0}^n a_k s^k$ is a monic polynomial of degree n with real coefficients, D denotes differentiation with respect to t, $\Delta_h : C(\mathbb{R}_h) \to C(\mathbb{R}_0)$ is the delay operator given by (1.2) (with $I = \mathbb{R}_h$), and Φ is a hysteresis operator. In problem (4.1), Φ maps $C(\mathbb{R}_h)$ into itself, whereas, in problem (4.2), Φ maps $C(\mathbb{R}_0)$ into itself.

A *solution* of (4.1) (respectively, (4.2)) on the interval $[-h, T)$ for some $0 < T \le \infty$ is a function $x \in C([-h, T))$ such that the restriction $x|_{[0,T)}$ is in $C^n([0, T))$ and satisfies (4.1a) (respectively, (4.2a)), and (4.1b) (respectively, (4.2b)) holds; a solution is said to be *global* if $T = \infty$.

Proposition 4.1. *If Φ satisfies* (N3) *and* (N4)*, then each of the initial-value problems* (4.1) *and* (4.2) *has a unique global solution (no finite-escape time).*

Proposition 4.1 is proved in Appendix A.

We proceed to consider two specific cases:

Case A, wherein p is assumed to be a Hurwitz polynomial;

Case B, wherein p is given by $p(s) = sq(s)$ for some Hurwitz polynomial q of degree $n-1$.

As we shall see, the latter case arises naturally in the context of feedback systems with integral control.

4.1. Case A

In the context of the initial-value problem (4.1), we have the following result.

Theorem 4.2. *Assume that every root of p has negative real part. Let $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ satisfy* (N1)−(N4) *with an associated constant λ, and let x be the unique global solution of* (4.1)*. If*

$$\inf_{\omega\in\mathbb{R}} \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}\omega h}/p(\mathrm{i}\omega)) > -1/\lambda, \tag{4.3}$$

then x has the following properties.

(i) *There exist constants $\beta = \beta(p, h, \lambda) < 0$ and $\gamma = \gamma(p, h, \lambda) > 0$ such that*

$$\|x\|_{W^{n,\infty}(\mathbb{R}_0)} + \|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} + \|x'\|_{W^{n-1,2}_\beta(\mathbb{R}_0)} + \|(\Phi(x))'\|_{L^2_\beta(\mathbb{R}_0)} \le \gamma\left(|\varphi(0)| + \sum_{k=1}^{n}|x_k| + \|\Phi(\varphi)\|_{C([-h,0])}\right). \tag{4.4}$$

(ii) $\lim_{t\to\infty} x^{(k)}(t) = 0,\ k = 1,\dots,n$, *the limits $x^\infty := \lim_{t\to\infty} x(t)$ and $\Phi^\infty := \lim_{t\to\infty}(\Phi(x))(t)$ exist, and all limits are approached at exponential rates; moreover,*

$$\Phi^\infty = -a_0 x^\infty. \tag{4.5}$$

Proof. Let $w : \mathbb{R}_0 \to \mathbb{R}$ and $v : \mathbb{R}_0 \to \mathbb{R}$ be the solutions of the initial-value problems

$$p(D)w = 0, \qquad w^{(k)}(0) = 0, \quad k = 0,\dots,n-2, \quad w^{(n-1)}(0) = 1$$

and

$$p(D)v = 0, \qquad v(0) = \varphi(0), \quad v^{(k)}(0) = x_k, \quad k = 1,\dots,n-1,$$

respectively. Let x be the unique global solution of (4.1) (which exists by virtue of Proposition 4.1). Viewing (4.1a) as the linear system $p(D)x + f = 0$, with forcing function $t \mapsto f(t) := (\Delta_h\Phi(x))(t)$, a routine calculation (an application of the variation of parameters formula for a linear n-th order system) shows that x satisfies

$$x(t) = v(t) - \int_0^t w(t-\tau)(\Delta_h\Phi(x))(\tau)\mathrm{d}\tau, \quad \forall\, t \ge 0. \tag{4.6}$$

Set

$$\rho(t) := v(t) - \begin{cases} \displaystyle\int_0^t w(t-\tau)(\Delta_h\Phi(\varphi))(\tau)\mathrm{d}\tau, & t \in [0,h] \\ \displaystyle\int_0^h w(t-\tau)(\Delta_h\Phi(\varphi))(\tau)\mathrm{d}\tau, & t \ge h \end{cases} \tag{4.7}$$

and denote by g the right shift of w by h:

$$g(t) := \begin{cases} 0, & t \in [0,h) \\ w(t-h), & t \ge h. \end{cases} \tag{4.8}$$

Note that $\rho \in C^{n-1}(\mathbb{R}_0) \cap C^n(\mathbb{R}_0 \setminus \{h\})$. Whilst the function $\rho^{(n-1)}$ is not differentiable at $t = h$, its left and right derivatives exist at this point, and so $\rho^{(n)}$ can be considered as a function in $C(\mathbb{R}_0 \setminus \{h\})$ with a jump discontinuity at $t = h$. Similarly, $g \in C^{n-2}(\mathbb{R}_0) \cap C^{n-1}(\mathbb{R}_0 \setminus \{h\})$ and $g^{(n-1)}$ can be considered as a function in $C(\mathbb{R}_0 \setminus \{h\})$ with a jump discontinuity at $t = h$.

Equation (4.6) can be rewritten as

$$x(t) = \rho(t) - (g * \Phi(x))(t), \quad \forall t \geq 0. \tag{4.9}$$

Note that (4.9) is of the form (3.1) (with $r_1 = 0,\ r_2 = \rho$). The key idea is to apply Theorem 3.1 to (4.9). Denoting the transfer function of the operator $u \mapsto g * u$ by $\mathbf{G}$, we have

$$\mathbf{G}(s) = (\mathcal{L}g)(s) = \frac{\mathrm{e}^{-sh}}{p(s)},$$

and so, by hypothesis (4.3), condition (3.2) of Theorem 3.1 holds. We proceed to show that the other hypotheses of Theorem 3.1 hold in the context of (4.9). Clearly, $\rho \in W^{1,1}_{\text{loc}}(\mathbb{R}_0)$ and $\rho(0) = \varphi(0)$. By the hypothesis on p, there exists $\alpha < 0$ such that the real part of each root of p is less than α, and it follows that both g and ρ' are in $L^2_\alpha(\mathbb{R}_0)$. All hypotheses of Theorem 3.1 are now in place.

To prove assertion (i), we first invoke Theorem 3.1 to infer the existence of constants $\beta \in (\alpha, 0)$ and $\gamma_1 > 0$ (dependent only on p, h and λ) such that

$$\begin{aligned}&\|x\|_{L^\infty(\mathbb{R}_0)} + \|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} + \|x'\|_{L^2_\beta(\mathbb{R}_0)} + \|(\Phi(x))'\|_{L^2_\beta(\mathbb{R}_0)} \\ &\quad \leq \gamma_1 \left(\|\rho'\|_{L^2_\beta(\mathbb{R}_0)} + |\varphi(0)| + |(\Phi(\varphi))(0)| \right).\end{aligned} \tag{4.10}$$

To establish (4.4), we proceed to estimate $\|x\|_{W^{n,\infty}(\mathbb{R}_0)}$, $\|x'\|_{W^{n-1,2}_\beta(\mathbb{R}_0)}$, $\|\rho\|_{W^{n,\infty}(\mathbb{R}_0)}$ and $\|\rho\|_{W^{n,2}_\beta(\mathbb{R}_0)}$ in terms of $\Phi(\varphi)$ and the initial conditions. Since $\beta \in (\alpha, 0)$, it follows that

$$\|\rho\|_{W^{n,\infty}(\mathbb{R}_0)} + \|\rho\|_{W^{n,2}_\beta(\mathbb{R}_0)} \leq \gamma_2 \left(|\varphi(0)| + \sum_{k=1}^{n-1} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right) \tag{4.11}$$

for some constant $\gamma_2 > 0$ (dependent only on p and h). Therefore, the right-hand side of (4.10) can be majorized by $\gamma_3(|\varphi(0)| + \sum_{k=1}^{n-1} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])})$ for some constant $\gamma_3 > 0$ (dependent only on p, h and λ). Consequently,

$$\begin{aligned}&\|x\|_{L^\infty(\mathbb{R}_0)} + \|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} + \|x'\|_{L^2_\beta(\mathbb{R}_0)} + \|(\Phi(x))'\|_{L^2_\beta(\mathbb{R}_0)} \\ &\quad \leq \gamma_3 \left(|\varphi(0)| + \sum_{k=1}^{n-1} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right).\end{aligned} \tag{4.12}$$

Differentiating (4.9), we obtain

$$x^{(k)} = \rho^{(k)} - g^{(k)} * \Phi(x), \quad k = 1, \ldots, n-1 \tag{4.13}$$

and

$$x^{(k)} = \rho^{(k)} - (\Phi(x))(0) g^{(k-1)} - g^{(k-1)} * (\Phi(x))', \quad k = 1, \ldots, n. \tag{4.14}$$

From (4.13), it follows that

$$\|x^{(k)}\|_{L^\infty(\mathbb{R}_0)} \leq \|\rho^{(k)}\|_{L^\infty(\mathbb{R}_0)} + \|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} \|g^{(k)}\|_{L^1(\mathbb{R}_0)}, \quad k = 1, \ldots, n-1.$$

Using (4.11) and (4.12), together with the fact that x solves (4.1), we obtain that

$$\|x^{(k)}\|_{L^\infty(\mathbb{R}_0)} \leq \gamma_4 \left(|\varphi(0)| + \sum_{k=1}^{n-1} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right), \quad k = 1, \ldots, n \tag{4.15}$$

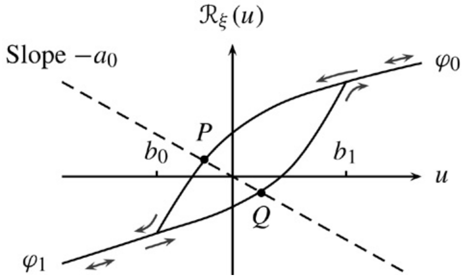

Fig. 9. Relay hysteresis, $(x^{\infty}, \Phi^{\infty}) \in \{P, Q\}$.

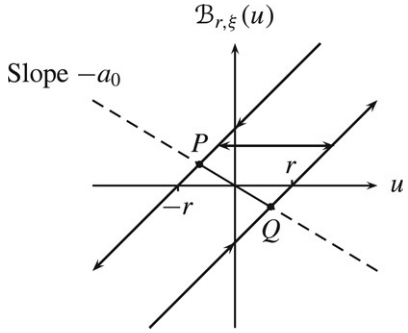

Fig. 10. Backlash hysteresis, $(x^{\infty}, \Phi^{\infty}) \in [P, Q]$.

for some constant $\gamma_4 > 0$ (dependent only on p, h and λ). Invoking (4.14), an argument similar to that leading to (4.15) gives

$$\|x'\|_{W_{\beta}^{n-1,2}(\mathbb{R}_0)} \leq \gamma_5 \left(|\varphi(0)| + \sum_{k=1}^{n-1} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right) \tag{4.16}$$

for some constant $\gamma_5 > 0$ (dependent only on p, h and λ). Combining (4.12), (4.15) and (4.16) yields the existence of a constant $\gamma > 0$ (dependent only on p, h and λ) such that (4.4) holds.

To prove assertion (ii), first note that

$$\rho^{(k)} \in L_{\alpha}^{\infty}(\mathbb{R}_0), \quad g^{(k-1)} \in L_{\alpha}^{2}(\mathbb{R}_0) \cap L_{\alpha}^{\infty}(\mathbb{R}_0), \quad k = 1, \ldots, n.$$

Combining this with the fact that $(\Phi(x))' \in L_{\beta}^{2}(\mathbb{R}_0)$ and invoking (4.14), yields

$$\lim_{t \to \infty} \mathrm{e}^{-\beta t} x^{(k)}(t) = 0, \quad k = 1, \ldots, n. \tag{4.17}$$

Again, using the fact that $(\Phi(x))' \in L_{\beta}^{2}(\mathbb{R}_0)$, we deduce that the limit $\Phi^{\infty} := \lim_{t \to \infty}(\Phi(x))(t)$ exists, is finite and is approached exponentially fast. Combining this with (4.17) and (4.1) and recalling that $a_0 \neq 0$ (by hypothesis on p), we conclude that $x(t)$ converges exponentially fast to a limit x^{∞} as $t \to \infty$. Therefore, $a_0 x^{\infty} + \Phi^{\infty} = \lim_{t \to \infty} (p(D)x + \Delta_h \Phi(x))(t) = 0$, whence (4.5). □

Remark 4.3. For many hysteresis operators, (4.5) can be used to obtain quantitative estimates of the limits x^{∞} and Φ^{∞}. For example, in the case of relay hysteresis, $(x^{\infty}, \Phi^{\infty})$ is one of at most two points of intersection of the graph of the nonlinearity and the line with slope $-a_0$ passing through the origin (see Fig. 9); in the case of backlash hysteresis (see Fig. 10), $x^{\infty} \in [-r/(1+a_0), r/(1+a_0)]$ and $\Phi^{\infty} \in [-a_0 r/(1+a_0), a_0 r/(1+a_0)]$; for the elastic–plastic operator, $x^{\infty} \in [-r/a_0, r/a_0]$ (see Fig. 11), whilst $\Phi^{\infty} \in [-r, r]$ (the latter does not provide additional information). ◇

Next, we consider the initial-value problem (4.2) (in which Φ maps $C(\mathbb{R}_0)$ into itself).

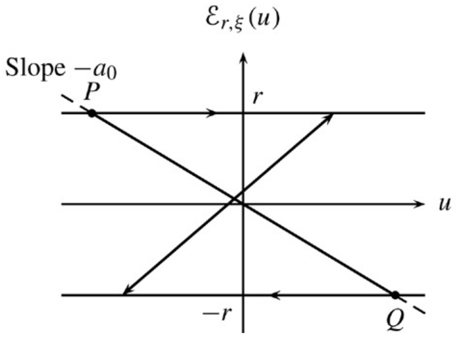

Fig. 11. Elastic–plastic hysteresis, $(x^\infty, \Phi^\infty) \in [P, Q]$.

Theorem 4.4. *Assume that every root of p has a negative real part. Let $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ satisfy* (N1)–(N4) *with an associated constant λ, and x be the unique global solution of* (4.2). *If* (4.3) *holds, then x has the following properties.*

(i) *There exist constants $\beta = \beta(p, h, \lambda) < 0$ and $\gamma = \gamma(p, h, \lambda) > 0$ such that*

$$\|x\|_{W^{n,\infty}(\mathbb{R}_0)} + \|\Phi(\Delta_h x)\|_{L^\infty([h,\infty))} + \|x'\|_{W^{n-1,2}_\beta(\mathbb{R}_0)} + \|(\Phi(\Delta_h x))'\|_{L^2_\beta([h,\infty))}$$
$$\leq \gamma \left(|\varphi(0)| + \sum_{k=1}^{n} |x_k| + \|\Phi(\Delta_h \varphi)\|_{C([0,h])} \right). \tag{4.18}$$

(ii) $\lim_{t\to\infty} x^{(k)}(t) = 0,\ k = 1, \ldots, n$, *the limits*

$$x^\infty := \lim_{t\to\infty} x(t) \quad \text{and} \quad \Phi^\infty := \lim_{t\to\infty} (\Phi(\Delta_h x))(t)$$

exist, and all limits are approached at exponential rate; moreover,

$$\Phi^\infty = -a_0 x^\infty.$$

Proof. Note that, by (2.6), the differential equation (4.2a) can be written in the form

$$p(D)x + \Delta_h \tilde{\Phi}(x) = 0, \tag{4.19}$$

where $\tilde{\Phi} : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is the operator defined by (2.5). Hence, an application of Proposition 2.2 and Theorem 4.2 establishes the claim. □

Remark 4.5. (i) Consider (4.1a) with forcing (or disturbance) $f \in C(\mathbb{R}_0)$, *i.e.*,

$$\left.\begin{array}{l} p(D)x + \Delta_h \Phi(x) = f, \\ x|_{[-h,0]} = \varphi \in C([-h,0]), \quad x^{(k)}(0) = x_k,\ k = 1, \ldots, n-1 \end{array}\right\} \tag{4.20}$$

An analysis of the proof of Proposition 4.1 shows that, under the same hypotheses, its assertions remains valid in the context of the initial-value problem (4.20). Moreover, if the assumptions of Theorem 4.2 hold and if $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0)$ with $f' \in L^2_\alpha(\mathbb{R}_0)$ for some $\alpha < 0$, then the conclusions of Theorem 4.2 remain valid for the initial-value problem (4.20) with $\beta \in (\alpha, 0)$ and $\gamma > 0$ independent of f and the initial conditions, provided that the term $\gamma(\|f'\|_{L^2_\beta(\mathbb{R}_0)} + |f(0)|)$ is added to the right-hand side of (4.4) and (4.5) is replaced by $\Phi^\infty = f^\infty - a_0 x^\infty$, where $f^\infty := \lim_{t\to\infty} f(t)$. This can readily be shown by modifying the proof of Theorem 4.2, the only modification being that Theorem 3.1 is now applied in the context of the equation $x = w * f + \rho - g * \Phi(x)$ (which replaces (4.9)). Finally, under the same assumptions on f, a similarly modified version of Theorem 4.4 holds for the initial-value problem (4.2) with forcing f.

(ii) A remark analogous to Remark 4.3 applies to Theorem 4.4. ◇

4.2. Case B

Next, we consider the initial-value problems (4.1) and (4.2) in the case where p takes the following form:

$$p(s) = sq(s), \quad \text{where } q(s) := \sum_{k=0}^{n-1} a_{k+1} s^k. \tag{4.21}$$

The next result relates to the initial-value problem (4.1).

Theorem 4.6. *Assume that p is of the form (4.21) and that every root of q has a negative real part. Let $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ be a hysteresis operator satisfying (N1)–(N5), with associated constant λ, and assume that $0 \in \operatorname{clos} \operatorname{NVS} \Phi$. If*

$$\inf_{\omega \in \mathbb{R}^*} \operatorname{Re}\,(\mathrm{e}^{-\mathrm{i}\omega h}/p(\mathrm{i}\omega)) > -1/\lambda, \tag{4.22}$$

then the unique global solution x of (4.1) has the following properties.

(i) $\lim_{t\to\infty} x^{(k)}(t) = 0,\ k = 1, \ldots, n$. *Furthermore, there exists a $\gamma = \gamma(p, h, \lambda) > 0$ such that*

$$\|x'\|_{W^{n-1,\infty}(\mathbb{R}_0)} + \|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} + \|(\Phi(x))'\|_{L^2(\mathbb{R}_0)} \le \gamma \left(\sum_{k=1}^{n} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right). \tag{4.23}$$

(ii) *If* (N6) *holds and* $0 \in \operatorname{int} \operatorname{NVS} \Phi$, *then x is bounded.*

(iii) *If* (N6) *holds,* $0 \in \operatorname{int} \operatorname{NVS} \Phi$ *and Φ is Lipschitz continuous, then*

$$\lim_{t\to\infty} \operatorname{dist}\,(x(t), L_\Phi(0)) = 0,$$

where $L_\Phi(0)$ is given by (2.9).

(iv) *If Φ satisfies a positive generalized sector condition with thresholds $\tau_1 \le 0, \tau_2 \ge 0$ and sector bound $\sigma > 0$, then*

$$\|x\|_{L^\infty(\mathbb{R}_0)} \le \max \left\{ -\tau_1,\ \tau_2,\ \frac{\gamma}{\sigma} \left(\sum_{k=1}^{n} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])} \right) \right\} \tag{4.24}$$

and

$$\lim_{t\to\infty} \operatorname{dist}\,(x(t), [\tau_1, \tau_2]) = 0. \tag{4.25}$$

Moreover, if $\tau_1 < 0$ and $\tau_2 > 0$, then there exists $T \ge 0$ such that

$$x(t) \in [\tau_1, \tau_2], \quad \forall\, t \ge T. \tag{4.26}$$

Proof. Recall that x satisfies the integral equation (4.9). In view of (4.21), g can be written as

$$g = \theta * g_0,$$

where g_0 is such that

$$(\mathfrak{L} g_0)(s) = \frac{\mathrm{e}^{-sh}}{q(s)} =: \mathbf{G}(s).$$

Hence

$$x(t) = \rho(t) - \int_0^t (g_0 * \Phi(x))(\tau)\mathrm{d}\tau, \quad \forall\, t \ge 0, \tag{4.27}$$

where ρ is given in (4.7). Note that (4.27) is of the form (3.4) (with $r_1 = 0$, $r_2 = \rho$), and that $\mathbf{G}$ is the transfer function of the operator $u \mapsto g_0 * u$. The key idea is to apply Theorem 3.2 to (4.27). To this end, we note that, trivially, $\rho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0)$, $\rho' \in L^2(\mathbb{R}_0)$ and $\rho(0) = \varphi(0)$. Furthermore, $\mathbf{G}(0) = 1/a_1 > 0$ and, by (4.22),

$$\inf_{\omega \in \mathbb{R}^*} \operatorname{Re} \mathbf{G}(\mathrm{i}\omega)/(\mathrm{i}\omega) > -1/\lambda,$$

showing that (3.5) holds. By Theorem 3.2, we may now infer the existence of a constant $\gamma_1 > 0$ (dependent only on p and λ) such that

$$\|\Phi(x)\|_{L^\infty(\mathbb{R}_0)} + \|(\Phi(x))'\|_{L^2(\mathbb{R}_0)} \le \gamma_1 \left(\|\rho'\|_{L^2(\mathbb{R}_0)} + |(\Phi(\varphi))(0)|\right). \tag{4.28}$$

Furthermore,

$$\|\rho'\|_{L^2(\mathbb{R}_0)} \le \gamma_2 \left(\sum_{k=1}^{n} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])}\right) \tag{4.29}$$

for some $\gamma_2 > 0$ (dependent only on p, h and λ).

Differentiating (4.27) and invoking (4.2), we obtain

$$x^{(k)} = \rho^{(k-1)} - g^{(k-1)} * \Phi(x), \quad k = 1, \ldots, n-1. \tag{4.30}$$

The convergence

$$\lim_{t\to\infty} x^{(k)}(t) = 0, \quad k = 1, \ldots, n$$

and the estimate

$$\|x'\|_{W^{n-1,\infty}(\mathbb{R}_0)} \le \gamma_3 \left(\sum_{k=1}^{n} |x_k| + \|\Phi(\varphi)\|_{C([-h,0])}\right) \tag{4.31}$$

for some $\gamma_3 > 0$ (dependent only on p, h and λ) follow from (4.30) via arguments similar to those used in the proof of Theorem 4.2. Combining (4.28), (4.29) and (4.31), we arrive at (4.23).

Assertion (ii) is an immediate consequence of Theorem 3.2. To prove assertion (iii), note that x is bounded (by (ii)) and $\lim_{t\to\infty}(\Phi(x))(t) = 0$ (by (i)): the claim now follows from part (iii) of Proposition 2.3. Finally, to prove assertion (iv), note that, by the positive generalized sector condition,

$$x(t) \notin [\tau_1, \tau_2] \Rightarrow |x(t)| \le \frac{1}{\sigma}|(\Phi(x))(t)|,$$

which, in conjunction with (4.23), gives (4.24). Furthermore, since $(\Phi(x))(t)$ converges to 0 as $t \to \infty$, (4.25) and (4.26) also follow. □

Remark 4.7. (i) It is not difficult to show that, with suitable modifications, Theorems 4.2 and 4.6 remain true for multiple point delays and distributed delays. More precisely, assume that (4.1a) is replaced by

$$(p(D)x)(t) + \int_{-h}^{0} (\Phi(x))(t+\tau)\mathrm{d}v(\tau) = 0, \quad \forall\, t \ge 0,$$

where v is a function of bounded variation on $[-h, 0]$. If the term $\mathrm{e}^{-\mathrm{i}\omega h}$ in (4.3) and (4.22) is replaced by the term $\int_{-h}^{0} \mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}v(\tau)$, then the assertions of Proposition 4.1, Theorems 4.2 and 4.6 remain valid.

(ii) Part (iv) of Theorem 4.6 (and (4.25) in particular) quantifies the asymptotic behaviour of the solution under the assumption that the hysteresis operator Φ satisfies a positive generalized sector condition. By contrast, part (iii) applies to many hysteresis nonlinearities which are bounded or have sub-linear growth (in the sense that their envelopes are bounded or have slower than linear growth): these nonlinearities do not satisfy a positive generalized sector condition. Prototype examples of such hysteresis nonlinearities are "saturated" backlash operators of the form $\mathcal{B}_{r,\xi} \circ \Sigma_l$ or $\Sigma_l \circ \mathcal{B}_{r,\xi}$ with $l > 0$, where the saturation operator $\Sigma_l : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is defined by

$$(\Sigma_l(u))(t) = \begin{cases} u(t), & \text{if } |u(t)| \le l, \\ l\,\mathrm{sign}(u(t)), & \text{if } |u(t)| > l. \end{cases}$$

If Φ is such a hysteresis operator which, in addition, is Lipschitz continuous, satisfies (N6) and has 0 in its numerical value set, then the solution approaches the set $L_\Phi(0) = \{z \in \mathbb{R} \mid \varphi_-(z) \le 0 \le \varphi_+(z)\}$, where (φ_-, φ_+) is the envelope of Φ.

(iii) Theorem 4.6 asserts the boundedness of x, provides quantitative information on its asymptotic behaviour, and establishes that $(\Phi(x))(t) \to 0$ as $t \to \infty$. However, under the hypotheses of Theorem 4.6, it is not generally the case that $x(t)$ converges as $t \to \infty$: a counterexample is provided below. ◇

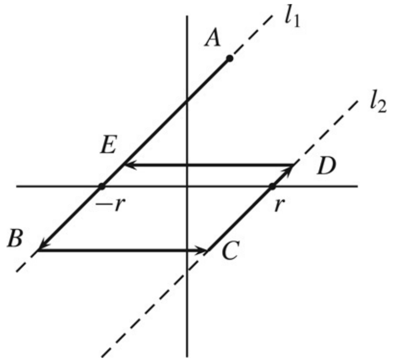

Fig. 12. Non-convergence of x.

Example 4.8 (*Non-convergence of the Solution of* (4.1) *under the Hypotheses of Theorem* 4.6). Consider the delay-free ($h = 0$) second-order system with backlash hysteresis $\mathcal{B}_{r,\xi}$ (as defined in Section 2.2):

$$x''(t) + ax'(t) + \big(\mathcal{B}_{r,\xi}(x)\big)(t) = 0, \quad x(0) = x_0, \quad x'(0) = x_1, \quad a > 0. \tag{4.32}$$

As previously remarked, $\mathcal{B}_{r,\xi}$ satisfies (N1)–(N6), with $\lambda = 1$ in (N3). Note further that $L_{\mathcal{B}_{r,\xi}}(0) = [-r, r]$ (recall definition (2.9)). In this case, $p(s) = s^2 + as$, and so hypothesis (4.3) holds, provided that $a > 1$. Therefore, assuming $a > 1$, Theorem 4.6 implies, *inter alia*, that the (unique) solution of (4.32) is bounded and approaches the interval $[-r, r]$, whence $\Omega(x) \subset [-r, r]$, where $\Omega(x)$ denotes the ω-limit set of x. Moreover, $x'(t) \to 0$ as $t \to \infty$. However, it is not necessarily the case that $x(t)$ converges as $t \to \infty$, as will now be shown.

Consider the delay-free case $h = 0$ (in which case, (4.3) is equivalent to the condition $a > 1$), with initial conditions $\xi > 0$, $x_0 = \xi - r$ (this implies that $(\mathcal{B}_{r,\xi}(x))(0) = \xi$) and $x_1 \geq 0$. Let x be the unique solution of the corresponding initial-value problem. We claim that if $1 < a < 2$, then $x(t)$ does not converge as $t \to \infty$: in particular, we will show that $\Omega(x) = [-r, r]$. With reference to Fig. 12, let l_1 and l_2 be the lines (each of slope 1) passing through $(-r, 0)$ and $(r, 0)$, respectively. Observe that $A = (x(0), (\mathcal{B}_{r,\xi}(x))(0)) = (\xi - r, \xi) \in l_1$.

Define

$$t_1 = \inf\{t \geq 0 : (x(t), (\mathcal{B}_{r,\xi}(x))(t)) \notin l_1\}$$

and note that $t_1 > 0$. Then,

$$x''(t) + ax'(t) + x(t) + r = 0, \quad \forall\, t \in [0, t_1), \tag{4.33}$$

whence,

$$\left.\begin{aligned} x(t) &= -r + \mathrm{e}^{-at/2}(c_1 \sin \alpha t + c_2 \cos \alpha t) \\ x'(t) &= \mathrm{e}^{-at/2}\left(\Big(-\frac{ac_1}{2} - \alpha c_2\Big)\sin \alpha t + \Big(-\frac{ac_2}{2} + \alpha c_1\Big)\cos \alpha t\right) \end{aligned}\right\}, \quad \forall\, t \in [0, t_1), \tag{4.34}$$

where $\alpha = \sqrt{1 - (a/2)^2} > 0$, and the constants c_1 and c_2 are determined by the initial conditions: specifically, $c_2 = \xi > 0$ and $c_1 = (2x_1 + a\xi)/(2\alpha)$. Noting that $x'(t) \leq 0$ for all $t \in [0, t_1)$, we may conclude from the second of equations (4.34) that $t_1 < \infty$. Moreover, at time t_1, the following must hold: $x'(t_1) = 0$, $x''(t_1) \geq 0$. Therefore, $(\mathcal{B}_{r,\xi}(x))(t_1) = x(t_1) + r \leq 0$. Suppose that $x(t_1) = -r$. Then, by (4.34), we have

$$c_1 \sin \alpha t_1 + c_2 \cos \alpha t_1 = 0 = -c_2 \sin \alpha t_1 + c_1 \cos \alpha t_1$$

which implies that $c_1 = 0 = c_2$, contradicting the fact that $c_2 = \xi > 0$. Therefore

$$(\mathcal{B}_{r,\xi}(x))(t_1) = x(t_1) + r =: z_1 < 0.$$

Writing $B = (x(t_1), z_1)$ and with reference to Fig. 12, we may conclude that the solution is such that, on the interval $[0, t_1]$, the path AB is traced: from B (at time t_1), the solution x is such that the path BC is then followed, with generating equation

$$x''(t) + ax'(t) + z_1 = 0,$$

until time t_2 given by

$$t_2 := \inf\{t \geq t_1 : (x(t), z_1) \in l_2\} > t_1.$$

Consequently,

$$\left.\begin{aligned} x(t) &= x(t_1) + \frac{|z_1|}{a}\left[(t-t_1) - \frac{1}{a}\left(1 - \mathrm{e}^{-a(t-t_1)}\right)\right] \\ x'(t) &= \frac{|z_1|}{a}\left(1 - \mathrm{e}^{-a(t-t_1)}\right) \end{aligned}\right\}, \quad \forall\, t \in [t_1, t_2].$$

Re-applying the above argument *mutatis mutandis*, we may conclude that, from C (at time t_2 with $x'(t_2) > 0$), the solution is such that path CDE is followed, where $D = (x(t_3), (\mathcal{B}_{r,\xi}(x))(t_3))$ with $t_3 := \inf\{t \geq t_2 : (x(t), (\mathcal{B}_{r,\xi}(x))(t)) \notin l_2\}$ and $(\mathcal{B}_{r,\xi}(x))(t_3) > 0$. The above construction may be repeated indefinitely, from which, together with the fact that, by Theorem 4.6, $(\mathcal{B}_{r,\xi}(x))(t) \to 0$ as $t \to \infty$, we may conclude that $\Omega(x) = [-r, r]$.

The above analysis is readily extended to conclude that, with $1 < a < 2$, $\Omega(x) = [-r, r]$ for all initial conditions such that $(\mathcal{B}_{r,\xi}(x))(0) \neq 0$. Finally, we note that, with $(\mathcal{B}_{r,\xi}(x))(0) \neq 0$, the limit $\lim_{t\to\infty} x(t)$ may exist (equivalently, $\Omega(x)$ may be a singleton) in the case where $a \geq 2$. For example, assume $\xi > 0$, $x(0) = \xi - r$ (then $(\mathcal{B}_{r,\xi}(x))(0) = \xi > 0$) and $x'(0) = 0$. Then (4.33) again applies, whence

$$x(t) = \begin{cases} -r + \dfrac{\xi}{\alpha_2 - \alpha_1}(\alpha_2 \mathrm{e}^{-\alpha_1 t} - \alpha_1 \mathrm{e}^{-\alpha_2 t}) & \text{if } a > 2, \\ -r + \xi(1+t)\mathrm{e}^{-t} & \text{if } a = 2, \end{cases} \qquad \forall\, t \in [0, t_1),$$

where $\alpha_1 = (a/2) - \sqrt{(a/2)^2 - 1}$ and $\alpha_2 = (a/2) + \sqrt{(a/2)^2 - 1}$. In this case, $t_1 = \infty$ and $x(t) \to -r$ as $t \to \infty$. ◇

Finally, we focus on the initial-value problem (4.2). Writing the differential equation (4.2a) in the form (4.19) and using Proposition 2.2 and Theorem 4.6, we arrive at the following result.

Theorem 4.9. *Assume that p is of the form (4.21) and that every root of q has negative real part. Let $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ be a hysteresis operator satisfying* (N1)–(N5), *with associated constant λ, and assume that* $0 \in \operatorname{clos} \mathrm{NVS}\,\Phi$. *Let x be the unique global solution of (4.2). If (4.22) holds, then $\lim_{t\to\infty} x^{(k)}(t) = 0$, $k = 1, \ldots, n$, and there exists $\gamma = \gamma(p, h, \lambda) > 0$ such that*

$$\|x'\|_{W^{n-1,\infty}(\mathbb{R}_0)} + \|\Phi(\Delta_h x)\|_{L^\infty([h,\infty))} + \|(\Phi(\Delta_h x))'\|_{L^2([h,\infty))} \leq \gamma \left(\sum_{k=1}^{n} |x_k| + \|\Phi(\Delta_h \varphi)\|_{C([0,h])}\right).$$

Moreover, assertions (ii)–(iv) *of Theorem* 4.6 *hold, provided that $\|\Phi(\varphi)\|_{C([-h,0])}$ on the right-hand side of (4.24) is replaced by $\|\Phi(\Delta_h \varphi)\|_{C([0,h])}$.*

Remark 4.10. Theorems 4.6 and 4.9 can be generalized to cover forced versions of Eqs. (4.1) and (4.2) with the forcing $f \in C(\mathbb{R}_0) \cap L^2(\mathbb{R}_0)$ (cf. Remark 4.5, part (i)). ◇

Example 4.11 (*Integral Control Systems*). Systems of type (4.2), with monic polynomial p of the form (4.21), arise naturally in the context of integral control of linear systems with hysteresis and delay. To illustrate this, consider the feedback system – with measurement delay – shown in Fig. 13, where r is a constant reference signal, $\kappa > 0$ is the "integrator gain" and $\Psi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$ is a hysteresis operator and

$$q(s) = p(s)/s = \sum_{k=0}^{n-1} a_{k+1} s^k$$

is a monic Hurwitz polynomial (setting $n = 3$, $a_2 = a$ and $a_1 = b$, we recover the feedback system shown in Fig. 1). Here, the control objective is to cause the signal y to track asymptotically the reference signal, in the sense that $y(t) \to r$ as $t \to \infty$.

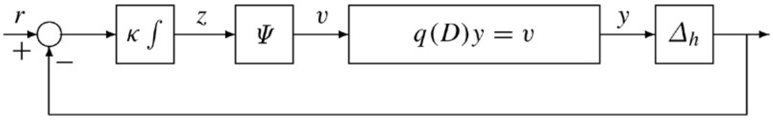

Fig. 13. Integral control of a system with input hysteresis and output delay.

The system illustrated in Fig. 13 is described by

$$\left.\begin{aligned} &q(D)y = (\Psi(z))(t),\ \ t \geq 0, \quad z'(t) = \kappa(r - y(t-h)),\ t \geq 0;\\ &y|_{[-h,0]} = \psi \in C([-h,0]), \quad y^{(k)}(0) = y_k,\ k = 1,\dots,n-2,\ \ z(0) = z_0. \end{aligned}\right\} \tag{4.35}$$

Therefore,

$$(p(D)z)(t+h) = \kappa[a_1 r - (\Psi z)(t)] \quad \forall\, t \geq 0.$$

Introducing the hysteresis operator $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$, $u \mapsto \kappa(\Psi(u) - a_1 r)$ and defining $x(\cdot) := z(\cdot + h)$, we have

$$p(D)x(t) + (\Phi(\Delta_h x))(t) = 0, \quad t \geq 0. \tag{4.36}$$

Defining $\varphi \in C([-h,0])$ by

$$\varphi(t) := z_0 + \kappa \int_{-h}^{t} [r - \psi(s)]\mathrm{d}s,$$

and writing

$$x_1 := \kappa[r - \psi(0)], \quad x_k := -\kappa y_{k-1}, \quad k = 2,\dots,n-1$$

we arrive at an initial-value problem of the form (4.2) with p satisfying (4.21). Now assume that Ψ satisfies (N1)–(N5) and $a_1 r \in \operatorname{clos} \mathrm{NVS}\,\Psi$. Then, an application of Theorem 4.9 to (4.36) shows that there exists a number $\kappa^* > 0$ such that for all $\kappa \in (0,\kappa^*)$, $x'(t)$ (and hence $z'(t)$) converges to zero as $t \to \infty$. Consequently, the tracking objective is achieved; that is

$$r = \lim_{t\to\infty} y(t-h) = \lim_{t\to\infty} y(t),$$

provided that $\kappa \in (0,\kappa^*)$. ◇

5. Second and third-order systems

This section is devoted to a specific study of systems (4.1) and (4.2), with the monic polynomial p assumed to be of degree two or three. We show that, in these special cases, the infima in (4.3) and (4.22) can be evaluated or estimated analytically in terms of the coefficients of p and the delay parameter $h \geq 0$.

Proposition 5.1. *Let the polynomial p be given by $p(s) = s^2 + a_1 s + a_0$, where $a_0 \geq 0,\ a_1 > 0$.*

(i) *If $a_0 = 0$, then, for all $h \geq 0$,*

$$\inf_{\omega\in\mathbb{R}^*} \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}\omega h}/p(\mathrm{i}\omega)) = -\frac{1 + a_1 h}{a_1^2}. \tag{5.1}$$

(ii) *If $a_0 > 0$ and $h = 0$, then*

$$\inf_{\omega\in\mathbb{R}} \mathrm{Re}\,(1/p(\mathrm{i}\omega)) = -\frac{1}{a_1^2 + 2a_1\sqrt{a_0}}. \tag{5.2}$$

(iii) *If $a_0 > 0$, then, for all $h > 0$,*

$$\inf_{\omega\in\mathbb{R}} \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}\omega h}/p(\mathrm{i}\omega)) \geq \begin{cases} -1/a_0, & \text{if } a_1^2 \geq 2a_0\\ -1/\left(a_1\sqrt{a_0 - (a_1/2)^2}\right), & \text{if } a_1^2 < 2a_0. \end{cases} \tag{5.3}$$

Remark 5.2. Note that the right-hand side of (5.3) does not depend on h, which makes it possible to apply Theorems 4.2 and 4.4 to second-order systems with arbitrary delays: for example, if $a_1^2 \geq 2a_0 > 0$, then (4.3) holds provided that the Lipschitz constant associated with the hysteresis operator is such that $\lambda < a_0$. ◇

Proposition 5.3. *Let the polynomial p be given by $p(s) = s(s^2 + a_2 s + a_1)$, where $a_1 > 0$, $a_2 > 0$.*
(i) *If $a_2^2 \geq 2a_1 - (a_1^2 h/(a_2 + a_1 h))$, then*

$$\inf_{\omega \in \mathbb{R}^*} \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}\omega}/p(\mathrm{i}\omega)) = -\frac{a_2 + a_1 h}{a_1^2} \tag{5.4}$$

(ii) *If $a_2^2 < 2a_1 - (a_1^2 h/(a_2 + a_1 h))$, then*

$$\inf_{\omega \in \mathbb{R}^*} \mathrm{Re}\,(\mathrm{e}^{-\mathrm{i}\omega h}/p(\mathrm{i}\omega)) \geq -\frac{1 + h\sqrt{a_1 - (a_2/2)^2}}{a_2(a_1 - (a_2/2)^2)}. \tag{5.5}$$

Proof of Propositions 5.1 and 5.3. Routine calculations yield (5.1) and (5.2); (5.4) is established in [15]. We proceed to prove (5.3) and (5.5). Let $a > 0$, $b > 0$, $h \geq 0$, and note that

$$\mathrm{Re}\,\frac{\mathrm{e}^{-\mathrm{i}h\omega}}{(\mathrm{i}\omega)^2 + a(\mathrm{i}\omega) + b} = \frac{(b - \omega^2)\cos\omega h - a\omega\sin\omega h}{(b - \omega^2)^2 + (a\omega)^2} =: u(\omega, h). \tag{5.6}$$

Apart from the specific case $h = 0$, it does not seem to be possible to analytically determine the infimum of u with respect to ω in (5.6). However, as we will see, it is not difficult to evaluate $\inf_{h \in \mathbb{R}_0} \inf_{\omega \in \mathbb{R}} u(\omega, h)$. To this end, recall that, for an arbitrary function of two variables, the order of taking infima can be interchanged:

$$\inf_{h \in \mathbb{R}_0} \inf_{\omega \in \mathbb{R}} u(\omega, h) = \inf_{(h,\omega) \in \mathbb{R}_0 \times \mathbb{R}} u(\omega, h) = \inf_{\omega \in \mathbb{R}} \inf_{h \in \mathbb{R}_0} u(\omega, h). \tag{5.7}$$

Observing that the numerator of u (defined in (5.6)) is equal to the inner product of the unit vector $(\cos\omega h, \sin\omega h)$ with the vector $(b - \omega^2, -a\omega)$, we obtain

$$\inf_{h \in \mathbb{R}_0} u(\omega, h) = -\frac{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}{(b - \omega^2)^2 + (a\omega)^2} = \frac{-1}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} \tag{5.8}$$

and

$$\sup_{h \in \mathbb{R}_0} u(\omega, h) = \frac{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}{(b - \omega^2)^2 + (a\omega)^2} = \frac{1}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}. \tag{5.9}$$

Invoking (5.7) and (5.8), a direct calculation yields

$$\inf_{h \in \mathbb{R}_0} \inf_{\omega \in \mathbb{R}} u(\omega, h) = \inf_{\omega \in \mathbb{R}} \frac{-1}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} = \begin{cases} -1/b, & \text{if } a^2 \geq 2b \\ -1\Big/\left(a\sqrt{b - (a/2)^2}\right), & \text{if } a^2 < 2b. \end{cases} \tag{5.10}$$

Setting $a = a_1$ and $b = a_0$ in (5.10) yields (5.3).

It remains only to establish (5.5). To this end, note that, by (5.9),

$$\sup_{\omega \in \mathbb{R}} \sup_{h \in \mathbb{R}_0} u(\omega, h) = \sup_{\omega \in \mathbb{R}} \frac{1}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} = -\inf_{\omega \in \mathbb{R}} \frac{-1}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}. \tag{5.11}$$

Consequently, by (5.10),

$$\sup_{\omega \in \mathbb{R}} \sup_{h \in \mathbb{R}_0} u(\omega, h) = \frac{1}{a\sqrt{b - (a/2)^2}}, \quad \text{if } a^2 < 2b. \tag{5.12}$$

Assume $a_2^2 < 2a_1 - (a_1^2 h/(a_2 + a_1 h))$. Setting $a = a_2$ and $b = a_1$, it follows from (5.12) that

$$\sup_{\omega \in \mathbb{R}} \sup_{\alpha \in \mathbb{R}_0} u(\omega, \alpha) = \frac{1}{a_2\sqrt{a_1 - (a_2/2)^2}}. \tag{5.13}$$

Writing

$$v(\omega, h) := \operatorname{Re} \frac{e^{-ih\omega}}{p(i\omega)} = \operatorname{Re} \frac{e^{-ih\omega}}{i\omega((i\omega)^2 + a_2 i\omega + a_1)}$$

we have

$$v(\omega, h) = v(\omega, 0) - \int_0^h u(\omega, \alpha) d\alpha \geq v(\omega, 0) - h \sup_{\alpha \in \mathbb{R}_0} u(\omega, \alpha), \quad \forall \omega \in \mathbb{R}^*, \forall h \in \mathbb{R}_0. \tag{5.14}$$

Now,

$$\inf_{\omega \in \mathbb{R}^*} v(\omega, 0) = \inf_{\omega \in \mathbb{R}^*} \frac{-a_2}{(\omega - a_1)^2 + (a_2 \omega)^2} = \frac{-1}{a_2(a_1 - (a_2/2)^2)},$$

which, together with (5.13) and (5.14), yields the result

$$\begin{aligned} \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}(e^{-i\omega h}/p(i\omega)) &= \inf_{\omega \in \mathbb{R}^*} v(\omega, h) \\ &\geq \inf_{\omega \in \mathbb{R}^*} v(\omega, 0) - h \sup_{\omega \in \mathbb{R}} \sup_{\alpha \in \mathbb{R}_0} u(\omega, \alpha) = -\frac{1 + h\sqrt{a_1 - (a_2/2)^2}}{a_2(a_1 - (a_2/2)^2)}. \quad \square \end{aligned}$$

Example 5.4 (*Integral Control Systems*). Consider again the prototype system in Fig. 1, a particular case of Example 4.11, with $q(s) = s^2 + as + b$ ($a > 0$ and $b > 0$), hysteresis operator Ψ, gain parameter $\kappa > 0$, delay $h \geq 0$, and constant reference signal $r \in \mathbb{R}$. For the purposes of illustration, assume that $\Psi = \mathcal{B}_{r,\xi}$, the backlash operator and so (N1)–(N6) holds. As in Example 4.11, we introduce the hysteresis $\Phi : C(\mathbb{R}_0) \to C(\mathbb{R}_0)$, $u \mapsto \kappa(\Psi(u) - br)$, for which (N1)–(N6) again hold (with Lipschitz constant $\lambda = \kappa$ in (N3)). Define

$$\gamma := \begin{cases} (a + bh)/b^2, & \text{if } a^2 \geq 2b - ((b^2 h)/(a + bh)) \\ \left(1 + h\sqrt{b - (a/2)^2}\right)/(a(b - (a/2)^2)), & \text{otherwise.} \end{cases}$$

In view of Proposition 5.3, (4.22) holds if $\kappa > 0$ is chosen sufficiently small so that $\kappa < 1/\gamma$. As in Example 4.11, we may now infer that the tracking objective $\lim_{t \to \infty} y(t) = r$ is achieved for every fixed value of gain $\kappa \in (0, 1/\gamma)$. ◇

Appendix A

This section is concerned with the proof of existence and uniqueness of global solutions to the initial-value problems (4.1) and (4.2) as claimed in Proposition 4.1. In the following, $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^n$. For $w \in C([-h, \alpha], \mathbb{R}^n)$ (with $h, \alpha \geq 0$) and $\gamma, \delta > 0$, analogous to (2.1), we define

$$\mathcal{C}(w; \delta, \gamma; \mathbb{R}^n) := \left\{ v \in C([-h, \alpha + \gamma], \mathbb{R}^n) : v|_{[-h,\alpha]} = w, \max_{t \in [\alpha, \alpha+\gamma]} \|v(t) - w(\alpha)\| \leq \delta \right\}.$$

Consider the following initial-value problem:

$$u'(t) = (F(u))(t), \quad t \geq t_0, \tag{A.1a}$$
$$u|_{[-h,t_0]} = u_0, \tag{A.1b}$$

where $h \geq 0$, $t_0 \geq 0$ and $u_0 \in C([-h, t_0], \mathbb{R}^n)$ (if $h = t_0 = 0$, then $C([-h, t_0], \mathbb{R}^n) = \mathbb{R}^n$). We assume that the operator $F : C(\mathbb{R}_h, \mathbb{R}^n) \to C(\mathbb{R}_0, \mathbb{R}^n)$ is causal and satisfies the following hypotheses:

(H1) For all $\alpha \geq 0$ and $w \in C([-h, \alpha], \mathbb{R}^n)$, there exist $\delta > 0, \gamma > 0$ and a function $f : [0, \gamma] \to \mathbb{R}_0$, with $f(0) = 0$, continuous at zero and such that for all $\varepsilon \in (0, \gamma]$

$$\int_\alpha^{\alpha+\varepsilon} \|(F(y))(\tau) - (F(z))(\tau)\| d\tau \leq f(\varepsilon) \max_{\tau \in [\alpha, \alpha+\varepsilon]} \|y(\tau) - z(\tau)\| \quad \forall y, z \in \mathcal{C}(w; \delta, \varepsilon; \mathbb{R}^n). \tag{A.2}$$

(H2) For all $\alpha > 0$ and $y \in C([-h, \alpha), \mathbb{R}^n)$, there exists $c > 0$ such that

$$\max_{\tau \in [-h,t]} \|(F(y))(\tau)\| \le c \left(1 + \max_{\tau \in [-h,t]} \|y(\tau)\|\right), \quad \forall\, t \in [0, \alpha).$$

Let I be an interval of the form $I = [-h, T]$ (with $t_0 < T < \infty$) or $I = [-h, T)$ (with $t_0 < T \le \infty$). A *solution* of the initial-value problem (A.1) on I is a function $u \in C(I, \mathbb{R}^n)$ such that u is continuously differentiable on $I \cap [t_0, \infty)$ and satisfies (A.1). Note that, for bounded I, F "localizes" to an operator mapping $C(I, \mathbb{R}^n)$ into $C(I \cap \mathbb{R}_0, \mathbb{R}^n)$; that is, $F(u)$ is well-defined when $u \in C(I, \mathbb{R}^n)$ (see Remark 1.1).

Lemma A.1. *For every $t_0 \ge 0$ and every $u_0 \in C([-h, t_0], \mathbb{R}^n)$, there exists a unique solution u of (A.1) defined on $\mathbb{R}_h$ (no finite escape-time).*

Remark A.2. In the proof of Proposition 4.1 at the end of this appendix, we will apply Lemma A.1 with $t_0 = 0$. However, to prove extended uniqueness (see Step 2 in the proof below), it is convenient, even in the case $t_0 = 0$, to have existence and uniqueness on a small interval (see Step 1 in the proof below) for any general $t_0 \ge 0$. For this reason, we consider the initial-value problem (A.1) for $t_0 \ge 0$. ⋄

Proof of Lemma A.1. We proceed in three steps.

Step 1. Existence and uniqueness on $[-h, t_0 + \varepsilon]$ for small $\varepsilon > 0$.

By (H1) (with $\alpha = t_0$ and $w = u_0$), there exist $\delta > 0$, $\gamma > 0$ and $f : [0, \gamma] \to \mathbb{R}_h$ with $f(0) = 0$, continuous at zero and such that, for all $\varepsilon \in (0, \gamma]$,

$$\int_{t_0}^{t_0+\varepsilon} \|(F(y))(\tau) - (F(z))(\tau)\| \mathrm{d}\tau \le f(\varepsilon) \max_{\tau \in [t_0, t_0+\varepsilon]} \|y(\tau) - z(\tau)\| \quad \forall\, y, z \in \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n). \tag{A.3}$$

Let Γ^ε, parameterized by $\varepsilon \in (0, \gamma]$, denote the operator defined on $C([-h, t_0 + \varepsilon], \mathbb{R}^n)$ by

$$(\Gamma^\varepsilon(x))(t) := \begin{cases} u_0(t), & t \in [-h, t_0] \\ u_0(t_0) + \displaystyle\int_{t_0}^{t} (F(x))(\tau)\mathrm{d}\tau, & t \in (t_0, t_0 + \varepsilon]. \end{cases}$$

Endowed with the metric

$$(y, z) \mapsto \max_{\tau \in [t_0, t_0+\varepsilon]} \|y(\tau) - z(\tau)\|,$$

$\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$ is a complete metric space. We will prove that for all $\varepsilon \in (0, \gamma]$ sufficiently small, Γ^ε is a strict contraction on $\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$. To show that

$$\Gamma^\varepsilon(\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)) \subset \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n),$$

for sufficiently small $\varepsilon > 0$, define $v \in (\mathbb{R}_h, \mathbb{R}^n)$ by

$$v(t) := \begin{cases} u_0(t), & -h \le t \le t_0 \\ u_0(t_0), & t > t_0. \end{cases}$$

If restricted to the interval $[-h, t_0 + \varepsilon]$, v belongs to $\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$. We do not distinguish notationally between v and its restriction. Let $y \in \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$ and $t \in [t_0, t_0 + \varepsilon]$. Then, invoking (A.3),

$$\begin{aligned} \|(\Gamma^\varepsilon(y))(t) - u_0(t_0)\| &\le \int_{t_0}^{t_0+\varepsilon} \|(F(y))(\tau)\| \mathrm{d}\tau \\ &\le \int_{t_0}^{t_0+\varepsilon} \|(F(y))(\tau) - (F(v))(\tau)\| \mathrm{d}\tau + \int_{t_0}^{t_0+\varepsilon} \|(F(v))(\tau)\| \mathrm{d}\tau \\ &\le f(\varepsilon) \max_{\tau \in [t_0, t_0+\varepsilon]} \|y(\tau) - u_0(t_0)\| + \tilde{f}(\varepsilon) \\ &\le f(\varepsilon)\delta + \tilde{f}(\varepsilon), \end{aligned}$$

where

$$\tilde{f}(\varepsilon) := \int_{t_0}^{t_0+\varepsilon} \|(F(v))(\tau)\| \mathrm{d}\tau.$$

Since $f(\varepsilon)$ and $\tilde{f}(\varepsilon)$ converge to 0 as $\varepsilon \downarrow 0$, it follows that, for all sufficiently small $\varepsilon > 0$,

$$\|(\Gamma^\varepsilon(y))(t) - u_0(t_0)\| \le \delta, \quad \forall\, t \in [t_0, t_0 + \varepsilon], \forall\, y \in \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n).$$

Thus $\Gamma^\varepsilon(\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)) \subset \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$, provided that $\varepsilon > 0$ is sufficiently small. Furthermore, using (A.3), we obtain that, for all $y, z \in \mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$ and for all $t \in [t_0, t_0 + \varepsilon]$,

$$\|(\Gamma^\varepsilon(y))(t) - (\Gamma^\varepsilon(z))(t)\| \le \int_{t_0}^{t_0+\varepsilon} \|(F(y))(\tau) - (F(z))(\tau)\| \mathrm{d}\tau \le f(\varepsilon) \max_{\tau \in [t_0, t_0+\varepsilon]} \|y(\tau) - z(\tau)\|.$$

Since $f(\varepsilon) < 1$ for all sufficiently small $\varepsilon > 0$, there exists $\varepsilon^* \in (0, \gamma]$ such that, for every $\varepsilon \in (0, \varepsilon^*]$, Γ^ε is a strict contraction on $\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$. Hence, Γ^ε has a unique fixed point in $\mathcal{C}(u_0; \delta, \varepsilon; \mathbb{R}^n)$ for every $\varepsilon \in (0, \varepsilon^*]$. In particular, there exists a unique solution u of (A.1) in $\mathcal{C}(u_0; \delta, \varepsilon^*; \mathbb{R}^n)$. However, at this point we cannot exclude the situation that there may exist other solutions on the interval $[-h, t_0 + \varepsilon^*]$ which do not belong to $\mathcal{C}(u_0; \delta, \varepsilon^*; \mathbb{R}^n)$. To establish uniqueness in $C([-h, t_0 + \varepsilon], \mathbb{R}^n)$ for sufficiently small $\varepsilon > 0$, we define

$$S := \{t \in [0, \varepsilon^*] : \|u(t_0 + t) - u_0(t_0)\| = \delta\}, \quad \varepsilon^{**} := \begin{cases} \inf S, & S \neq \emptyset \\ \varepsilon^*, & S = \emptyset. \end{cases}$$

It is clear that $\varepsilon^{**} \in (0, \varepsilon^*]$, and that u restricted to the interval $[-h, t_0 + \varepsilon^{**}]$ is the unique solution of (A.1) on this interval.

Step 2. Extended uniqueness.

Let $u_1 \in C([-h, t_1), \mathbb{R}^n)$ and $u_2 \in C([-h, t_2), \mathbb{R}^n)$ be solutions, where $t_1, t_2 > t_0$. Set $t_3 := \min\{t_1, t_2\}$. We claim that $u_1 = u_2$ on $[-h, t_3)$. Seeking a contradiction, suppose that the claim is false. Then

$$t^* := \inf\{t \in [-h, t_3) : u_1(t) \neq u_2(t)\} < t_3.$$

By Step 1, we have that $t^* > t_0$. Define $u_0^*(t) := u_1(t) = u_2(t)$ for all $t \in [-h, t^*]$. Again by Step 1 (applied in the case $t_0 = t^*$ and $u_0 = u_0^*$), there exists $\varepsilon \in (0, t_3 - t^*)$ and a unique solution $u \in C([-h, t^* + \varepsilon], \mathbb{R}^n)$. It follows that $u_1(t) = u_2(t) = u(t)$ on $[-h, t^* + \varepsilon]$, which contradicts the definition of t^*.

Step 3. Existence on $\mathbb{R}_h$.

Let $\mathcal{T}$ be the set of all $\tau > t_0$ such that there exists a solution u^τ of (A.1) on the interval $[-h, \tau)$. By Step 1, $\mathcal{T} \neq \emptyset$. Let $T := \sup \mathcal{T}$, and define a function $u : [-h, T) \to \mathbb{R}^n$ by setting

$$u(t) = u^\tau(t), \quad \text{for } t \in [-h, \tau), \text{ where } \tau \in \mathcal{T}.$$

By Step 2, the function u is well defined, *i.e.*, the definition of $u(t)$ for a particular value $t \in [-h, T)$ does not depend on the choice of $\tau \in \mathcal{T} \cap (t, \infty)$. Moreover, it is clear that u is a solution of (A.1) on the interval $[-h, T)$. It remains to show that $T = \infty$. Seeking a contradiction, suppose that $T < \infty$. Observing that

$$\max_{\tau \in [t_0, t]} \|u(\tau)\| \le \|u(t_0)\| + \int_{t_0}^{t} \max_{\sigma \in [t_0, \tau]} \|(F(u))(\sigma)\| \mathrm{d}\tau, \quad \forall\, t \in [t_0, T),$$

and invoking assumption (H2) (with $\alpha = T$ and $y = u$), we see that there exists a $c > 0$ such that

$$\max_{\tau \in [t_0, t]} \|u(\tau)\| \le \|u(t_0)\| + cT + c \int_{t_0}^{t} \max_{\sigma \in [t_0, \tau]} \|u(\sigma)\| \mathrm{d}\tau, \quad \forall\, t \in [t_0, T).$$

An application of Gronwall's lemma to the function $t \mapsto \max_{\tau \in [t_0, t]} \|u(\tau)\|$ now yields

$$\|u(t)\| \le \max_{\tau \in [t_0, t]} \|u(\tau)\| \le (\|u(t_0)\| + cT)\mathrm{e}^{c(T - t_0)}, \quad \forall\, t \in [t_0, T),$$

showing that u is bounded. Consequently, by (H2), $F(u)$ is bounded, which combined with

$$u(t) = u(t_0) + \int_{t_0}^{t} (F(u))(\tau)\mathrm{d}\tau, \quad \forall\, t \in [t_0, T),$$

implies that $\lim_{t\to T} u(t)$ exists. Setting

$$u_T(t) := \begin{cases} u(t) & \text{for } t \in [-h, T), \\ \lim\limits_{t\to T} u(t) & \text{for } t = T \end{cases}$$

and using Step 1 (with $t_0 = T$ and $u_0 = u_T$), shows that the solution u can be extended to a solution defined on $[-h, T+\varepsilon)$. Consequently, $\sup \mathcal{T} \geq T + \varepsilon$, contradicting the definition of T. □

For convenience, we record the following lemma, which is useful in the proof of Proposition 4.1.

Lemma A.3. *If $\Phi : C(\mathbb{R}_h) \to C(\mathbb{R}_h)$ is a hysteresis operator satisfying* (N3)*, then the following holds:*
(N3$'$) *There exists $\lambda > 0$ such that, for all $\alpha \geq -h$ and $w \in C([-h, \alpha])$, there exist numbers $\gamma, \delta > 0$ such that*

$$\max_{\tau\in[\alpha,\alpha+\varepsilon]} |(\Phi(u))(\tau) - (\Phi(v))(\tau)| \leq \lambda \max_{\tau\in[\alpha,\alpha+\varepsilon]} |u(\tau) - v(\tau)|, \quad \forall\, u, v \in \mathcal{C}(w; \delta, \gamma), \forall \varepsilon \in (0, \gamma]. \tag{A.4}$$

Proof. Let λ be the constant associated with (N3), let $\alpha \geq -h$ and let $w \in C([-h, \alpha])$. By (N3), there exist $\gamma, \delta > 0$ such that (2.2) holds. Letting $\varepsilon \in (0, \gamma]$ and invoking the operator Q_τ defined in (1.1), we see that, by (N3) and the causality of Φ,

$$\begin{aligned} \max_{\tau\in[\alpha,\alpha+\varepsilon]} |(\Phi(u))(\tau) - (\Phi(v))(\tau)| &\leq \max_{\tau\in[\alpha,\alpha+\gamma]} |(\Phi(Q_{\alpha+\varepsilon}u))(\tau) - (\Phi(Q_{\alpha+\varepsilon}v))(\tau)| \\ &\leq \lambda \max_{\tau\in[\alpha,\alpha+\gamma]} |(Q_{\alpha+\varepsilon}u)(\tau) - (Q_{\alpha+\varepsilon}v)(\tau)| \\ &= \lambda \max_{\tau\in[\alpha,\alpha+\varepsilon]} |u(\tau) - v(\tau)| \end{aligned}$$

for all $u, v \in \mathcal{C}(w; \delta, \gamma)$. □

Proof of Proposition 4.1. We prove the existence and uniqueness of global solutions of (4.1). The corresponding result for the initial-value problem (4.2) then follows by appealing to to Proposition 2.2, by reducing the latter problem to the former with the operator $\tilde{\Phi}$ (defined in (2.5)) replacing Φ (recall that (4.2a) can be written in the form (4.19)).

Set $u_k := x^{(k-1)}$, $k = 1, \ldots, n$, and rewrite (4.1) as

$$\left.\begin{aligned} &u_1' = u_2, \qquad u_1(t) = \varphi(t), \quad \forall t \in [-h, 0], \\ &u_2' = u_3, \qquad u_2(t) = x_1, \quad \forall\, t \in [-h, 0], \\ &\vdots \\ &u_{n-1}' = u_n, \qquad u_{n-1}(t) = x_{n-1}, \quad \forall t \in [-h, 0], \\ &u_n' = -\sum_{k=0}^{n-1} a_k u_{k+1} - \Delta_h \Phi(u_1), \qquad u_n(t) = x_n, \quad \forall t \in [-h, 0]. \end{aligned}\right\} \tag{A.5}$$

Defining the causal operator $F : C(\mathbb{R}_h, \mathbb{R}^n) \to C(\mathbb{R}_0, \mathbb{R}^n)$ by

$$F(y_1, \ldots, y_n) := \begin{pmatrix} y_2|_{\mathbb{R}_0} \\ \vdots \\ y_n|_{\mathbb{R}_0} \\ -\sum\limits_{k=0}^{n-1} a_k y_{k+1}|_{\mathbb{R}_0} - \Delta_h \Phi(y_1) \end{pmatrix},$$

the initial-value problem (A.5) can be written in the form (A.1) (with $t_0 = 0$). By Lemma A.1, it is sufficient to show that F satisfies (H1) and (H2). Invoking (N4), it follows trivially that (H2) holds. It remains to establish (H1). To this end, let $\gamma_1, \gamma_2 > 0$, be such that

$$\gamma_1 \sum_{k=1}^{n} |\xi_k| \le \|\xi\| \le \gamma_2 \sum_{k=1}^{n} |\xi_k|, \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

(such constants exists due to equivalence of norms on $\mathbb{R}^n$).

Let $\alpha \ge 0$ and $w \in C([-h, \alpha], \mathbb{R}^n)$ be arbitrary. It follows from (N3) and Lemma A.3 that there exist $\gamma > 0$ and $\delta > 0$ such that, for all $\varepsilon \in (0, \gamma]$ and all $y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathcal{C}(w; \delta, \gamma; \mathbb{R}^n)$,

$$\max_{\tau \in [\alpha, \alpha+\varepsilon]} |(\Phi(y_1))(\tau) - (\Phi(z_1))(\tau)| \le \lambda \max_{\tau \in [\alpha, \alpha+\varepsilon]} |y_1(\tau) - z_1(\tau)|. \tag{A.6}$$

Consequently, for all $\varepsilon \in (0, \gamma]$ and all $y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathcal{C}(w; \delta, \gamma; \mathbb{R}^n)$,

$$\begin{aligned}\int_{\alpha}^{\alpha+\varepsilon} \|(F(y))(\tau) - (F(z))(\tau)\| \mathrm{d}\tau &\le \gamma_2 \int_{\alpha}^{\alpha+\varepsilon} \left(\sum_{k=2}^{n} |y_k(\tau) - z_k(\tau)| \right. \\ &\quad \left. + \left| \sum_{k=0}^{n-1} a_k (y_{k+1}(\tau) - z_{k+1}(\tau)) + (\Delta_h \Phi(y_1))(\tau) - (\Phi(z_1))(\tau) \right| \right) \mathrm{d}\tau \\ &\le \gamma_2 \varepsilon \left(\gamma_1^{-1} \left(1 + \sum_{k=0}^{n-1} |a_k| \right) \max_{\tau \in [\alpha, \alpha+\varepsilon]} \|y(\tau) - z(\tau)\| + \lambda \max_{\tau \in [\alpha, \alpha+\varepsilon]} |y_1(\tau) - z_1(\tau)| \right) \\ &\le \gamma_1^{-1} \gamma_2 \varepsilon \left(1 + \sum_{k=0}^{n-1} |a_k| + \lambda \right) \max_{\tau \in [\alpha, \alpha+\varepsilon]} \|y(\tau) - z(\tau)\|,\end{aligned}$$

showing that (H1) holds with $f(\varepsilon) = \gamma_1^{-1} \gamma_2 \varepsilon (1 + \sum_{k=0}^{n-1} |a_k| + \lambda)$. $\square$

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