Math. Control Signals Systems (2000) 13: 293–317

Mathematics of Control, Signals, and Systems

Time-Varying and Adaptive Discrete-Time Low-Gain Control of Infinite-Dimensional Linear Systems with Input Nonlinearities*

H. Logemann† and E. P. Ryan†

Abstract. Discrete-time low-gain control strategies are presented for tracking constant reference signals for infinite-dimensional, discrete-time, power-stable, linear systems subject to input nonlinearities. Both non-adaptive (but time-varying) and adaptive controls are considered. The discrete-time results are applied in the development of sampled-data integral control for infinite-dimensional, continuous-time, exponentially stable, regular, linear systems with input nonlinearities.

Key words. Discrete-time systems, Infinite-dimensional systems, Integral control, Input nonlinearities, Regular systems, Robust tracking, Sampled-data control.

1. Introduction

The paper extends a sequence [LR], [LRT1], [LRT2] of recent results pertaining to integral control of infinite-dimensional systems subject to input nonlinearities. Underpinning these results are generalizations of the well-known principle (see, for example, [D], [L2] and [M]) that closing the loop around a stable, linear, finite-dimensional, continuous-time, single-input single-output plant, with transfer function $\mathbf{G}_c$ compensated by a pure integral controller k/s, will result in a stable closed-loop system that achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $\mathbf{G}_c(0)k > 0$. (Therefore, under the above assumptions on the plant, the problem of tracking constant reference signals reduces to that of tuning the gain parameter k. This so-called "tuning regulator theory" [D] has been successfully applied in process control (see [CSW] and [L1]).) If we denote by $\mathscr{S}_c$ the class of single-input $(u(t))$, single-output $(y(t))$, continuous-time, infinite-dimensional, regular, exponentially stable, linear systems, and by $\mathscr{N}$ the set of globally Lipschitz non-decreasing functions $\varphi : \mathbb{R} \to \mathbb{R}$, then in [LRT1] it is shown that the above principle remains valid if

* Date received: October 1, 1998. Date revised: September 24, 1999. This work was supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/L78086).

† Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England. {hl,epr}@maths.bath.ac.uk.

the plant to be controlled is a class $\mathscr{S}_c$ system (with transfer function $\mathbf{G}_c$) subject to an input nonlinearity φ of class $\mathscr{N}$: more precisely, if $\mathbf{G}_c(0) > 0$ and if the constant reference signal r is feasible (in the sense that $[\mathbf{G}_c(0)]^{-1}r$ is in the closure of the image of φ), then there exists $k^* > 0$ such that, for all $k \in (0, k^*)$, the output $y(t)$ of the closed-loop system converges to r as $t \to \infty$. Let k^{**} denote the supremum of all such k^*. In [LRT1], it is shown that $k^{**} \geq \kappa^*/\lambda$, where $\lambda > 0$ is a Lipschitz constant for φ and κ^* denotes the supremum of all numbers $\kappa > 0$ such that

$$1 + \kappa \operatorname{Re} \frac{\mathbf{G}_c(s)}{s} \geq 0, \qquad \text{for all } s \text{ with } \operatorname{Re} s > 0.$$

Lower bounds and formulae for κ^* in terms of plant data are provided in [LRT2]. In general, k^{**} is a function of the plant data and so, in the presence of uncertainty, may fail to be computable. In such circumstances, it is natural to consider integral controllers with time-varying gain $k(\cdot)$ such that $k(t) \downarrow 0$ as $t \to \infty$ as well as adaptive gain strategies. Such considerations form the focus of [LR] wherein, in particular, the efficacy of the simple adaptive integral control strategy

$$\left.\begin{aligned} u(t) &= u_0 + \int_0^t k(\tau)[r - y(\tau)]\, d\tau, \qquad k(t) = \frac{1}{l(t)}, \\ \dot{l}(t) &= |r - y(t)|, \qquad l(0) = l_0 > 0, \end{aligned}\right\} \tag{1}$$

is established. In different contexts, the problem of tuning the integrator gain adaptively has been widely addressed elsewhere: see, for example, [C2], [MD1] and [MD2] for the finite-dimensional case (with input constraints treated in [MD2]), and [LT1]–[LT3] for the linear infinite-dimensional case.

The contribution of the present paper is twofold:

(i) a treatment of discrete-time low-gain control of infinite-dimensional, discrete-time, power-stable, linear systems subject to input nonlinearities of class $\mathscr{N}$, providing the discrete-time analogues of the results in [LR] on continuous-time systems;
(ii) application of the discrete-time results in the development of a sampled-data counterpart of the strategy (1) for adaptive control of continuous-time systems of class $\mathscr{S}_c$ subject to input nonlinearities of class $\mathscr{N}$.

With respect to (i), whilst the structure of the discrete-time analysis parallels that of the continuous-time analysis in [LR], there are several points where these analyses differ in an essential manner. With reference to (ii), the sampled-data results constitute the main contribution of the paper. In the derivation of these results, the discrete-time theory plays a central rôle.

2. Discrete-Time Low-Gain Control

The problem of tracking constant reference signals $r \in \mathbb{R}$ is first addressed in a context of single-input ($u_n \in \mathbb{R}$), single-output ($y_n \in \mathbb{R}$), discrete-time (time domain $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), linear systems having a nonlinearity φ in the input channel and

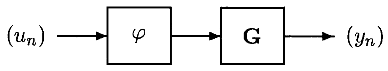

Fig. 1. The generic discrete-time system.

evolving on a real Banach space X (with norm $\|\cdot\|$):

$$x_{n+1} = Ax_n + B\varphi(u_n), \qquad x_0 \in X, \tag{2a}$$

$$y_n = Cx_n + D\varphi(u_n). \tag{2b}$$

Here $A \in \mathscr{B}(X, X) =: \mathscr{B}(X)$, $B \in \mathscr{B}(\mathbb{R}, X)$, $C \in \mathscr{B}(X, \mathbb{R})$ and $D \in \mathbb{R}$, where $\mathscr{B}(Y, Z)$ denotes the space of linear bounded operators from a Banach space Y to a Banach space Z. By tracking $r \in \mathbb{R}$ for a system of form (2), we mean the construction of an input sequence (u_n) such that $(\varphi(u_n))$ is bounded and $y_n \to r$ as $n \to \infty$. A system of the form (2) is called *power-stable* if A is power-stable, that is, there exist $M \geq 1$ and $\theta \in (0, 1)$ such that

$$\|A^n\| \leq M\theta^n, \qquad \forall n \in \mathbb{N}_0,$$

where $\|\cdot\|$ denotes the operator norm on $\mathscr{B}(X)$ induced by the norm $\|\cdot\|$ on X. Henceforth, we assume A is power-stable.

The generic system to be controlled is depicted schematically in Fig. 1, wherein the transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(\cdot) = C(\cdot I - A)^{-1}B + D$$

and is assumed to satisfy $\mathbf{G}(1) > 0$. In summary, the class of underlying quadruples (A, B, C, D) is denoted

$$\mathscr{S} := \{(A, B, C, D) | A \text{ power-stable}, \mathbf{G}(1) = C(I - A)^{-1}B + D > 0\}. \tag{3}$$

The input nonlinearities to be considered are those functions φ of class $\mathscr{N}$ defined below.

Definition 2.1. Let $\mathscr{N}$ be the class of functions $\varphi : \mathbb{R} \to \mathbb{R}$ with the properties: (a) φ is monotone non-decreasing; (b) φ satisfies a global Lipschitz condition (with Lipschitz constant λ), that is, for some $\lambda \geq 0$, $|\varphi(u) - \varphi(v)| \leq \lambda|u - v|$ for all u, $v \in \mathbb{R}$.

As an example of $\varphi \in \mathscr{N}$, consider the input nonlinearity in Fig. 2.

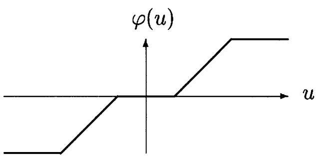

Fig. 2. Nonlinearity with saturation and dead zone.

The Clarke [C1] directional derivative $\varphi^o(u;v)$ of $\varphi \in \mathcal{N}$ at u in direction v is given by

$$\limsup_{\substack{w\to u\\ h\downarrow 0}} \frac{\varphi(w+hv)-\varphi(w)}{h}.$$

Define $\varphi^-(\cdot) := -\varphi^o(\cdot;-1)$ (if φ is C^1 with derivative φ', then $\varphi^- \equiv \varphi'$). A point $u \in \mathbb{R}$ is said to be a *critical point* (and $\varphi(u)$ is said to be a *critical value*) of φ if $\varphi^-(u) = 0$.

We record the following technicality for later use.

Proposition 2.2. *Let $\varphi \in \mathcal{N}$, with Lipschitz constant $\lambda > 0$. Let $(u_n) \subset \mathbb{R}$ and define $(\delta_n) \subset [0,\lambda]$ by*

$$\delta_n := \begin{cases} \dfrac{\varphi(u_{n+1})-\varphi(u_n)}{u_{n+1}-u_n} & \text{if} \quad u_{n+1} \neq u_n, \\ \lambda & \text{if} \quad u_{n+1} = u_n. \end{cases}$$

If (u_n) is convergent and its limit is not a critical point of φ, then there exist constants $N \in \mathbb{N}$ and $\mu > 0$ such that

$$\delta_n \geq \mu, \qquad \forall n \geq N.$$

Proof. Let (u_n) be convergent with limit u a non-critical point of φ (and so $\varphi^-(u) > 0$). Seeking a contradiction, suppose that (δ_n) is not bounded away from zero. Then, extracting a subsequence if necessary, we may assume that $\delta_n \to 0$ as $n \to \infty$. Write $h_n = u_{n+1} - u_n$. Clearly, the sequence (h_n) must contain a subsequence (h_{n_k}) with either (i) $h_{n_k} > 0$ for all $k \in \mathbb{N}$, or (ii) $h_{n_k} < 0$ for all $k \in \mathbb{N}$. We will show that each case leads to a contradiction.

Case (i): *Assume $h_{n_k} > 0$ for all k.* Then we have the contradiction:

$$\begin{aligned} 0 &= \lim_{k\to\infty} \delta_{n_k} = \lim_{k\to\infty} \frac{\varphi(u_{n_k}+h_{n_k})-\varphi(u_{n_k})}{h_{n_k}} \\ &\geq \liminf_{\substack{w\to u\\ h\downarrow 0}} \frac{\varphi(w+h)-\varphi(u)}{h} = -\limsup_{\substack{w\to u\\ h\downarrow 0}} \frac{\varphi(w)-\varphi(w+h)}{h} \\ &= -\limsup_{\substack{v\to u\\ h\downarrow 0}} \frac{\varphi(v-h)-\varphi(v)}{h} = -\varphi^o(u;-1) = \varphi^-(u) > 0. \end{aligned}$$

Case (ii): *Assume $h_{n_k} < 0$ for all k.* Then we have the contradiction:

$$\begin{aligned} 0 &= \lim_{k\to\infty} \delta_{n_k} = \lim_{k\to\infty} \frac{\varphi(u_{n_k}+h_{n_k})-\varphi(u_{n_k})}{h_{n_k}} \\ &= -\lim_{k\to\infty} \frac{\varphi(u_{n_k}-|h_{n_k}|)-\varphi(u_{n_k})}{|h_{n_k}|} \\ &\geq -\limsup_{\substack{w\to u\\ h\downarrow 0}} \frac{\varphi(w-h)-\varphi(w)}{h} = -\varphi^o(u;-1) = \varphi^-(u) > 0. \end{aligned}$$

■

Denote by $r \in \mathbb{R}$ the value of the constant reference signal to be tracked by the output y_n. The following condition plays a central rôle: $[\mathbf{G}(1)]^{-1} r \in \operatorname{clos}(\operatorname{im}\varphi)$. Reference values r satisfying this condition are referred to as *feasible*. The next proposition demonstrates that feasibility of r is a necessary condition for solvability of the tracking problem.

Proposition 2.3. *Let $(A, B, C, D) \in \mathscr{S}$, $\varphi \in \mathscr{N}$ and $r \in \mathbb{R}$. If there exists a sequence (u_n) such that $(\varphi(u_n))$ is bounded and the solution $n \mapsto x_n$ of (2) has the property*

$$\lim_{n\to\infty} [Cx_n + D\varphi(u_n)] = r,$$

then $\varphi_r := [\mathbf{G}(1)]^{-1} r \in \operatorname{clos}(\operatorname{im}\varphi)$.

Proof. Using the identity $(I - A)^{-1}A = (I - A)^{-1} - I$ it follows, from (2), that

$$\begin{aligned} C(I - A)^{-1}x_{n+1} &= C(I - A)^{-1}x_n - Cx_n + C(I - A)^{-1}B\varphi(u_n) \\ &= C(I - A)^{-1}x_n - y_n + \mathbf{G}(1)\varphi(u_n), \end{aligned}$$

and hence

$$C(I - A)^{-1}x_n = C(I - A)^{-1}x_0 + \mathbf{G}(1)\sum_{j=0}^{n-1}(\varphi(u_j) - [\mathbf{G}(1)]^{-1}y_j). \tag{4}$$

The function φ is continuous, and so $\operatorname{clos}(\operatorname{im}\varphi)$ is a closed interval (which may be unbounded). Seeking a contradiction, suppose that $[\mathbf{G}(1)]^{-1}r \notin \operatorname{clos}(\operatorname{im}\varphi)$. Then

$$d := \operatorname{dist}([\mathbf{G}(1)]^{-1}r, \operatorname{clos}(\operatorname{im}(\varphi)) > 0$$

and, since $\lim_{j\to\infty} y_j = r$, it follows that for all sufficiently large j, the term $\varphi(u_j) - [\mathbf{G}(1)]^{-1}y_j$ has constant sign and is bounded away from zero. Consequently, the right-hand side of (4) is unbounded as $n \to \infty$. However, by the power-stability of A and boundedness of $(\varphi(u_n))$, the left-hand side of (4) remains bounded as $n \to \infty$, yielding a contradiction. ∎

2.1. Time-Varying Gain

With a view to tracking feasible reference values $r \in \mathbb{R}$, we investigate control action of the form

$$u_{n+1} = u_n + k_n(r - Cx_n - D\varphi(u_n)),$$

where $(k_n) \subset \mathbb{R}$ is a suitably chosen sequence of gain parameters, which leads to the following nonlinear system of difference equations:

$$x_{n+1} = Ax_n + B\varphi(u_n), \qquad x_0 \in X, \tag{5a}$$

$$u_{n+1} = u_n + k_n(r - Cx_n - D\varphi(u_n)), \qquad u_0 \in \mathbb{R}. \tag{5b}$$

The main result of this section is contained in Theorem 2.6. The essence of this theorem is the assertion that if r is feasible and (k_n) decreases monotonically to zero and approaches zero sufficiently slowly, then both (x_n) and $(\varphi(u_n))$ converge.

In particular, the limit of $(\varphi(u_n))$ is $[\mathbf{G}(1)]^{-1}r$, thereby ensuring asymptotic tracking of r.

Lemma 2.4. *Let $(A, B, C, D) \in \mathscr{S}$, $\varphi \in \mathscr{N}$ with Lipschitz constant $\lambda > 0$ and $r \in \mathbb{R}$. Let $(k_n) \subset \mathbb{R}_+$ be a monotone sequence with $k_n \downarrow 0$ as $n \to \infty$. For all $(x_0, u_0) \in X \times \mathbb{R}$, the solution $n \mapsto (x_n, u_n)$ of* (5) *satisfies*

(a) $\lim_{n\to\infty} \varphi(u_n)$ *exists and is finite,*
(b) $\lim_{n\to\infty} x_n = (I - A)^{-1}B\varphi^*$, *where* $\varphi^* := \lim_{n\to\infty} \varphi(u_n)$.

Proof. Introduce new variables

$$z_n := x_n - (I - A)^{-1}B\varphi(u_n), \qquad v_n := \varphi(u_n) - \varphi_r,$$

where $\varphi_r := [\mathbf{G}(1)]^{-1}r$. Define the sequence $(\delta_n) \subset [0, \lambda]$ (as in Proposition 2.2) by

$$\delta_n := \begin{cases} \dfrac{\varphi(u_{n+1}) - \varphi(u_n)}{u_{n+1} - u_n} & \text{if} \quad u_{n+1} \neq u_n, \\ \lambda & \text{if} \quad u_{n+1} = u_n. \end{cases}$$

Using the identity $A(I - A)^{-1} = (I - A)^{-1} - I$, a straightforward calculation yields

$$\left.\begin{aligned} z_{n+1} &= Az_n + k_n\delta_n(I - A)^{-1}B(\mathbf{G}(1)v_n + Cz_n), \\ z_0 &= x_0 - (I - A)^{-1}B\varphi(u_0), \end{aligned}\right\} \tag{6a}$$

$$v_{n+1} = v_n - k_n\delta_n(\mathbf{G}(1)v_n + Cz_n), \qquad v_0 = \varphi(u_0) - \varphi_r. \tag{6b}$$

By the power-stability of A there exist constants $M \geq 1$ and $\theta \in (0, 1)$ such that

$$\|A^n\| \leq M\theta^n, \qquad \forall n \in \mathbb{N}_0.$$

Defining $\gamma_0 := M/(1 - \theta)$, we have

$$\gamma_0 = M\sum_{n=0}^{\infty}\theta^n \geq M\sqrt{\sum_{n=0}^{\infty}\theta^{2n}}. \tag{7}$$

Combining (6a), (7) and the fact that $(\delta_n) \subset [0, \lambda]$ with a routine estimate based on elementary properties of the convolution product of sequences we obtain, for all $m, N \in \mathbb{N}$,

$$\begin{aligned} \left(\sum_{n=N}^{m+N}\|z_n\|^2\right)^{1/2} &\leq \gamma_0\|z_N\| + \gamma_0\gamma_1\mathbf{G}(1)\left(\sum_{n=N}^{m+N}k_n^2\delta_n^2v_n^2\right)^{1/2} \\ &\quad + \gamma_0\gamma_1\|C\|\left(\sum_{n=N}^{m+N}k_n^2\delta_n^2\|z_n\|^2\right)^{1/2} \\ &\leq \gamma_0\|z_N\| + \gamma_0\gamma_1\sqrt{\lambda}\mathbf{G}(1)\left(\sum_{n=N}^{m+N}k_n^2\delta_nv_n^2\right)^{1/2} \\ &\quad + k_N\gamma_0\gamma_1\lambda\|C\|\left(\sum_{n=N}^{m+N}\|z_n\|^2\right)^{1/2}, \end{aligned} \tag{8}$$

where $\gamma_1 := \|(I - A)^{-1}B\|$. Choosing N sufficiently large so that

$$\alpha_0 := 1 - k_N\gamma_0\gamma_1\lambda\|C\| > 0,$$

it follows from (8) that

$$\left(\sum_{n=N}^{m+N} \|z_n\|^2\right)^{1/2} \leq \alpha_1\|z_N\| + \alpha_2\left(\sum_{n=N}^{m+N} k_n^2\delta_n v_n^2\right)^{1/2}, \qquad \forall m \in \mathbb{N}, \tag{9}$$

where

$$\alpha_1 := \gamma_0/\alpha_0, \qquad \alpha_2 := \gamma_0\gamma_1\sqrt{\lambda}\mathbf{G}(1)/\alpha_0.$$

As a consequence

$$\begin{aligned}\sum_{n=N}^{m+N} |k_n\delta_n v_n|\,\|z_n\| &\leq \left(\sum_{n=N}^{m+N} k_n^2\delta_n^2 v_n^2\right)^{1/2}\left(\sum_{n=N}^{m+N} \|z_n\|^2\right)^{1/2}\\ &\leq \alpha_3\|z_N\|\left(\sum_{n=N}^{m+N} k_n\delta_n v_n^2\right)^{1/2} + \alpha_4\sum_{n=N}^{m+N} k_n^2\delta_n v_n^2,\end{aligned} \tag{10}$$

where $\alpha_3 := \alpha_1\sqrt{k_0\lambda}$ and $\alpha_4 := \alpha_2\sqrt{\lambda}$. It follows from (6b) that, for all $m \in \mathbb{N}$,

$$v_{m+1+N}^2 = v_N^2 - 2\sum_{n=N}^{m+N} k_n\delta_n v_n(\mathbf{G}(1)v_n + Cz_n) + \sum_{n=N}^{m+N} k_n^2\delta_n^2(\mathbf{G}(1)v_n + Cz_n)^2, \tag{11}$$

whence

$$\begin{aligned}v_{m+1+N}^2 \leq v_N^2 &- 2\mathbf{G}(1)\sum_{n=N}^{m+N} k_n\delta_n v_n^2 + 2\|C\|\sum_{n=N}^{m+N} |k_n\delta_n v_n|\,\|z_n\|\\ &+ 2[\mathbf{G}(1)]^2\lambda\sum_{n=N}^{m+N} k_n^2\delta_n v_n^2 + 2k_0^2\lambda^2\|C\|^2\sum_{n=N}^{m+N} \|z_n\|^2.\end{aligned} \tag{12}$$

Combining (9), (10) and (12), there exist constants $\beta_1, \beta_2, \beta_3 > 0$ such that, for all $m \in \mathbb{N}$,

$$\begin{aligned}v_{m+1+N}^2 \leq v_N^2 + \beta_1\|z_N\|^2 &+ \beta_2\|z_N\|\left(\sum_{n=N}^{m+N} k_n\delta_n v_n^2\right)^{1/2}\\ &+ \sum_{n=N}^{m+N}\{\beta_3 k_n - 2\mathbf{G}(1)\}k_n\delta_n v_n^2.\end{aligned} \tag{13}$$

Since $k_n \downarrow 0$ as $n \to \infty$,

$$\beta_3 k_n - 2\mathbf{G}(1) \leq -\mathbf{G}(1) < 0 \qquad \text{for all } n \text{ sufficiently large.}$$

Consequently, by (13),

$$\sum_{n=0}^{\infty} k_n\delta_n v_n^2 < \infty. \tag{14}$$

Using (9) and (10) it follows that

$$\sum_{n=0}^{\infty} \|z_n\|^2 < \infty, \qquad \sum_{n=0}^{\infty} |k_n \delta_n v_n| \, \|z_n\| < \infty. \tag{15}$$

Combining (11), (14) and (15) we see that there exists a number $v \in [0, \infty)$ such that

$$\lim_{m \to \infty} v_m^2 = v. \tag{16}$$

If $v = 0$, then $\lim_{n\to\infty} v_n = 0$ and so $\lim_{n\to\infty} \varphi(u_n) = \varphi^* = \varphi_r$. Now consider the remaining case $v > 0$. Since $k_n \downarrow 0$ as $n \to \infty$, we may conclude from (6b) that

$$\lim_{n \to \infty} |v_{n+1} - v_n| = 0. \tag{17}$$

Since $v \neq 0$, (16) and (17) imply that v_n is of constant sign for all n sufficiently large. Thus, $\lim_{n\to\infty} v_n$ exists and takes one of the values $\pm\sqrt{v}$. We have now shown that the sequence (v_n) converges, whence assertion (a) of the theorem. Assertion (b) then follows by the power-stability of A. ∎

Lemma 2.5. *Let* $(A, B, C, D) \in \mathscr{S}$ *and* $\varphi \in \mathscr{N}$ *with Lipschitz constant* $\lambda > 0$. *Let* $r \in \mathbb{R}$ *be such that* $\varphi_r := [\mathbf{G}(1)]^{-1} r \in \operatorname{clos}(\operatorname{im}\varphi)$. *Let* $(k_n) \subset \mathbb{R}_+$ *be a bounded sequence and such that*

$$K_n := \sum_{j=0}^{n} k_j \to \infty \qquad \text{as} \quad n \to \infty.$$

For $(x_0, u_0) \in X \times \mathbb{R}$, *let* $n \mapsto (x_n, u_n)$ *be the solution of* (5). *If* $\lim_{n\to\infty} \varphi(u_n)$ *exists and is finite, then the following statements hold*:

(a) $\lim_{n\to\infty} \varphi(u_n) = \varphi_r$,
(b) $\lim_{n\to\infty} x_n = (I - A)^{-1} B \varphi_r$,
(c) $\lim_{n\to\infty} y_n = r$, *where* $(y_n) = (C x_n + D\varphi(u_n))$,
(d) *if* $\varphi_r \in \operatorname{im}\varphi$, *then*

$$\lim_{n\to\infty} \operatorname{dist}(u_n, \varphi^{-1}(\varphi_r)) = 0,$$

(e) *if* $\varphi_r \in \operatorname{int}(\operatorname{im}\varphi)$, *then* (u_n) *is bounded.*

Proof. By hypothesis, there exists $\varphi^* \in \mathbb{R}$ such that $\lim_{n\to\infty} \varphi(u_n) = \varphi^*$. The essence of the proof of assertions (a)–(e) is to show that $\varphi^* = \varphi_r = [\mathbf{G}(1)]^{-1} r$.

Seeking a contradiction, suppose that $\varphi^* \neq \varphi_r$. By the power-stability of A, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (C x_n + D\varphi(u_n)) = C(I - A)^{-1} B \varphi^* + D\varphi^* = \mathbf{G}(1)\varphi^*.$$

Let $N \in \mathbb{N}$ be such that

$$|y_n - \mathbf{G}(1)\varphi^*| \leq \tfrac{1}{2}\mathbf{G}(1)|\varphi_r - \varphi^*|, \qquad \forall n \geq N.$$

Noting that $r - y_n = \mathbf{G}(1)(\varphi_r - \varphi^*) - (y_n - \mathbf{G}(1)\varphi^*)$, it follows from (5b) that

$$-\tfrac{1}{2}k_n\mathbf{G}(1)|\varphi_r - \varphi^*| \leq u_{n+1} - u_n - k_n\mathbf{G}(1)(\varphi_r - \varphi^*) \leq \tfrac{1}{2}k_n\mathbf{G}(1)|\varphi_r - \varphi^*|, \qquad \forall n \geq N.$$

If $\varphi_r > \varphi^*$, then

$$\tfrac{1}{2}k_n\mathbf{G}(1)(\varphi_r - \varphi^*) \le u_{n+1} - u_n, \qquad \forall n \ge N,$$

whence

$$u_{m+1+N} \ge u_N + \tfrac{1}{2}\mathbf{G}(1)(\varphi_r - \varphi^*)\sum_{j=N}^{m+N} k_j = \alpha_0 + \alpha_1 K_{m+N}, \qquad \forall m \in \mathbb{N},$$

for some constants α_0, α_1 with $\alpha_1 > 0$. Since $K_n \to \infty$ as $n \to \infty$, we conclude that $u_n \to \infty$ as $n \to \infty$, whence the contradiction:

$$\varphi_r \le \sup\varphi = \lim_{n\to\infty}\varphi(u_n) = \varphi^* < \varphi_r.$$

If $\varphi_r < \varphi^*$, then

$$u_{n+1} - u_n \le -\tfrac{1}{2}k_n\mathbf{G}(1)|\varphi_r - \varphi^*|, \qquad \forall n \ge N,$$

and so

$$u_{m+1+N} \le \alpha_2 - \alpha_3 K_{m+N}, \qquad \forall m \in \mathbb{N},$$

for some constants α_2, α_3 with $\alpha_3 > 0$. Therefore, $u_n \to -\infty$ as $n \to \infty$ which yields the contradiction:

$$\varphi_r \ge \inf\varphi = \lim_{n\to\infty}\varphi(u_n) = \varphi^* > \varphi_r.$$

We may now conclude that $\varphi^* = \varphi_r$, which is assertion (a) of the lemma. By the power-stability of A, assertions (b) and (c) follow immediately.

To prove assertion (d), let $\varphi_r \in \operatorname{im}\varphi$ and suppose that the claim is false. Then there exists a subsequence of (u_n), which we do not relabel, and $\varepsilon > 0$ such that

$$\operatorname{dist}(u_n, \varphi^{-1}(\varphi_r)) \ge \varepsilon, \qquad \forall n. \tag{18}$$

If the sequence (u_n) is bounded, then we may assume without loss of generality that it converges to a finite limit u^*. By the continuity of φ and assertion (a), $\varphi(u^*) = \varphi_r$ and so $u^* \in \varphi^{-1}(\varphi_r)$ which contradicts (18). Now consider the case of unbounded (u_n). Since $\varphi_r \in \operatorname{im}\varphi$, there exists $u \in \mathbb{R}$ such that $\varphi(u) = \varphi_r$. By the unboundedness of (u_n) and extracting a subsequence if necessary, we may assume that either (i) $u_n \to +\infty$ as $n \to \infty$, or (ii) $u_n \to -\infty$ as $n \to \infty$. By the monotonicity of φ and assertion (a), in case (i) we have $\sup\varphi = \varphi_r = \varphi(u) = \varphi(u_n)$ for all n sufficiently large and so $u_n \in \varphi^{-1}(\varphi_r)$ for all n sufficiently large, contradicting (18), whilst in case (ii) we have $\inf\varphi = \varphi_r = \varphi(u) = \varphi(u_n)$ for all n sufficiently large and so $u_n \in \varphi^{-1}(\varphi_r)$ for all n sufficiently large, again contradicting (18). This establishes assertion (d).

Now assume $\varphi_r \in \operatorname{int}(\operatorname{im}\varphi)$ and, for contradiction, suppose that assertion (e) is false. Then either $\limsup_{n\to\infty} u_n = +\infty$ or $\liminf_{n\to\infty} u_n = -\infty$. The former case implies that $\sup\varphi = \varphi_r$ $(= \lim_{n\to\infty}\varphi(u_n))$ whilst the latter case implies that $\inf\varphi = \varphi_r$. Both cases contradict the hypothesis $\varphi_r \in \operatorname{int}(\operatorname{im}\varphi)$. ■

Theorem 2.6. *Let $(A, B, C, D) \in \mathcal{S}$ and $\varphi \in \mathcal{N}$ with Lipschitz constant $\lambda > 0$. Let $r \in \mathbb{R}$ be such that $\varphi_r := [\mathbf{G}(1)]^{-1}r \in \operatorname{clos}(\operatorname{im}\varphi)$. Let $(k_n) \subset \mathbb{R}_+$ be a monotone* *sequence with the properties that* $k_n \downarrow 0$ *as* $n \to \infty$ *and*

$$K_n := \sum_{j=0}^{n} k_j \to \infty \qquad as \quad n \to \infty.$$

For all $(x_0, u_0) \in X \times \mathbb{R}$, *the solution* $n \mapsto (x_n, u_n)$ *of* (5) *satisfies*

(a) $\lim_{n\to\infty} \varphi(u_n) = \varphi_r$,
(b) $\lim_{n\to\infty} x_n = (I - A)^{-1} B\varphi_r$,
(c) $\lim_{n\to\infty} y_n = r$, *where* $(y_n) = (Cx_n + D\varphi(u_n))$,
(d) *if* $\varphi_r \in \operatorname{im}\varphi$, *then*

$$\lim_{n\to\infty} \operatorname{dist}(u_n, \varphi^{-1}(\varphi_r)) = 0,$$

(e) *if* $\varphi_r \in \operatorname{int}(\operatorname{im}\varphi)$, *then* (u_n) *is bounded*,
(f) *if* $\varphi_r \in \operatorname{im}\varphi$ *is not a critical value of* φ, *then the convergence in* (a) *and* (b) (*and hence in* (c)) *is of order* ρ^{-K_n} *for some* $\rho > 1$ (*in the sense that the sequences* $(\rho^{K_n}[\varphi(u_n) - \varphi_r])$, $(\rho^{K_n}[x_n - (I - A)^{-1}B\varphi_r])$ *and* $(\rho^{K_n}(r - y_n))$ *are bounded*).

Proof. Assertions (a)–(e) follow immediately from Lemmas 2.4 and 2.5. It remains only to establish assertion (f).

Introduce new variables

$$z_n := \rho^{K_n}(x_n - (I - A)^{-1}B\varphi(u_n)), \qquad v_n := \rho^{K_n}(\varphi(u_n) - \varphi_r).$$

It suffices to prove the boundedness of the sequences (z_n) and (v_n). As in the proof of Lemma 2.4, define the sequence $(\delta_n) \subset [0, \lambda]$ by

$$\delta_n := \begin{cases} \dfrac{\varphi(u_{n+1}) - \varphi(u_n)}{u_{n+1} - u_n} & \text{if} \quad u_{n+1} \neq u_n, \\ \lambda & \text{if} \quad u_{n+1} = u_n. \end{cases}$$

Since $\varphi_r \in \operatorname{im}\varphi$ is not a critical value, $\varphi^{-1}(\varphi_r)$ is a singleton $\{u_r\}$, with $\varphi^-(u_r) > 0$ and, in view of assertion (d), $u_n \to u_r$ as $n \to \infty$. By Proposition 2.2, there exists $\mu > 0$ such that

$$0 < \mu \leq \delta_n, \qquad \forall n.$$

A straightforward calculation yields

$$\left.\begin{aligned} z_{n+1} &= \Delta_n z_n + k_n \sigma_n (I - A)^{-1} B(\mathbf{G}(1)v_n + Cz_n), \\ z_0 &= \rho^{k_0}(x_0 - (I - A)^{-1}B\varphi(u_0)), \end{aligned}\right\} \tag{19a}$$

$$v_{n+1} = \rho^{k_{n+1}} v_n - k_n \sigma_n (\mathbf{G}(1)v_n + Cz_n), \qquad v_0 = \rho^{k_0}(\varphi(u_0) - \varphi_r), \tag{19b}$$

where

$$\sigma_n := \rho^{k_{n+1}}\delta_n \quad \text{and} \quad \Delta_n := \rho^{k_{n+1}} A, \qquad \forall n.$$

Clearly,

$$0 < \mu \leq \delta_n \leq \sigma_n \leq \nu := \rho^{k_0}\lambda.$$

By (19a), for all $N \geq 0$,

$$z_{m+N} = \left(\prod_{n=N}^{m+N-1} \Delta_n\right) z_N + \sum_{n=N}^{m+N-1} \left(\prod_{i=n+1}^{m+N-1} \Delta_i\right) k_n \sigma_n (I - A)^{-1} B(\mathbf{G}(1) v_n + C z_n), \qquad \forall m \in \mathbb{N}, \tag{20}$$

wherein we adopt the notational convention $\prod_{i=m+N}^{m+N-1} \Delta_i := I$.

By the power-stability of A, there exist $M \geq 1$ and $\theta \in (0, 1)$ such that $\|A^n\| \leq M\theta^n$ for all $n \in \mathbb{N}_0$. Observe that, for $p \in \mathbb{N}$ with $p > n$,

$$\left\| \prod_{j=n+1}^{p} \Delta_j \right\| = \left\| \left(\prod_{j=n+1}^{p} \rho^{k_{j+1}} \right) A^{p-n} \right\| = \|\rho^{k_{n+2}+\cdots+k_{p+1}} A^{p-n}\| \leq M(\rho^{k_0}\theta)^{p-n}. \tag{21}$$

Choose $\rho > 1$ sufficiently close to 1 such that

$$\text{(a)}\ \rho^{k_0}\theta < 1 \quad \text{and} \quad \text{(b)}\ 4 \ln \rho \leq \mu \mathbf{G}(1). \tag{22}$$

Then, by (22a),

$$\gamma_0 := \frac{M}{1 - \rho^{k_0}\theta} = M \sum_{n=0}^{\infty} (\rho^{k_0}\theta)^n \geq M \sqrt{\sum_{n=0}^{\infty} (\rho^{k_0}\theta)^{2n}}. \tag{23}$$

Set $\gamma_1 := \|(I - A)^{-1} B\|$ and choose N sufficiently large so that

$$\text{(a)}\ \alpha_0 := 1 - k_N \gamma_0 \gamma_1 \nu \|C\| > 0 \quad \text{and} \quad \text{(b)}\ \frac{\rho^{2k_n} - 1}{k_n} \leq 4 \ln \rho, \qquad \forall n \geq N. \tag{24}$$

(Note that, since $k_n \downarrow 0$ as $n \to \infty$, we have $\lim_{n\to\infty} (\rho^{2k_n} - 1)/k_n = 2 \ln \rho$ and so there exists $N \in \mathbb{N}$ such that (24b) holds.)

Using (20), (21), (23) and (24a), a calculation similar to that leading to (9) yields

$$\left(\sum_{n=N}^{m+N} \|z_n\|^2 \right)^{1/2} \leq \alpha_1 \|z_N\| + \alpha_2 \left(\sum_{n=N}^{m+N} k_n^2 v_n^2 \right)^{1/2}, \qquad \forall m \in \mathbb{N}, \tag{25}$$

where $\alpha_1 := \gamma_0/\alpha_0$ and $\alpha_2 := \gamma_0 \gamma_1 \nu \mathbf{G}(1)/\alpha_0$. It follows that

$$\sum_{n=N}^{m+N} |k_n v_n|\, \|z_n\| \leq \alpha_1 \sqrt{k_0} \|z_N\| \left(\sum_{n=N}^{m+N} k_n v_n^2 \right)^{1/2} + \alpha_2 \sum_{n=N}^{m+N} k_n^2 v_n^2, \qquad \forall m \in \mathbb{N}. \tag{26}$$

By (19b),

$$v_{n+1}^2 - v_n^2 = (\rho^{2k_{n+1}} - 1) v_n^2 - 2\rho^{k_{n+1}} k_n \sigma_n v_n (\mathbf{G}(1) v_n + C z_n) + k_n^2 \sigma_n^2 (\mathbf{G}(1) v_n + C z_n)^2, \tag{27}$$

whence

$$v_{n+1}^2 - v_n^2 \leq \left(\frac{\rho^{2k_{n+1}} - 1}{k_{n+1}} \right) k_n v_n^2 - 2\mu \mathbf{G}(1) k_n v_n^2 + 2\rho^{k_0} \nu \|C\|\, |k_n v_n|\, \|z_n\| + 2\nu^2 [\mathbf{G}(1)]^2 k_n^2 v_n^2 + 2k_0^2 \nu^2 \|C\|^2 \|z_n\|^2.$$

By (22b), (24b), (25) and (26), there exist constants $\beta_1, \beta_2, \beta_3 > 0$ such that, for all $m \in \mathbb{N}$,

$$v_{m+1+N}^2 \le v_N^2 + \beta_1 \|z_N\|^2 + \beta_2 \|z_N\| \left(\sum_{n=N}^{m+N} k_n v_n^2 \right)^{1/2} + \sum_{n=N}^{m+N} [\beta_3 k_n - \mu \mathbf{G}(1)] k_n v_n^2. \tag{28}$$

Since $k_n \downarrow 0$ as $n \to \infty$,

$$\beta_3 k_n - \mu \mathbf{G}(1) \le -\mu \mathbf{G}(1)/2 < 0 \qquad \text{for all } n \text{ sufficiently large.}$$

Consequently, by (28),

$$\text{(a)} \ \sum_{n=0}^{\infty} k_n v_n^2 < \infty \quad \text{and} \quad \text{(b)} \ (v_n) \text{ is bounded.} \tag{29}$$

By (25) and (29a), $\sum_{n=0}^{\infty} \|z_n\|^2 < \infty$ and so $z_n \to 0$ as $n \to \infty$. This completes the proof. ■

2.2. *Adaptive Gain*

Whilst Theorem 2.6 identifies conditions under which the tracking objective is achieved through the use of a monotone gain sequence, the resulting control strategy is somewhat unsatisfactory insofar as the gain sequence is selected *a priori:* no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain sequence. In particular, let the gain sequence (k_n) be generated by the law

$$k_n = \frac{1}{l_n}, \qquad l_{n+1} = l_n + |r - y_n|, \qquad l_0 > 0, \tag{30}$$

which yields the feedback system

$$x_{n+1} = A x_n + B \varphi(u_n), \qquad x_0 \in X, \tag{31a}$$

$$u_{n+1} = u_n + l_n^{-1}(r - C x_n - D\varphi(u_n)), \qquad u_0 \in \mathbb{R}, \tag{31b}$$

$$l_{n+1} = l_n + |r - C x_n - D\varphi(u_n)|, \qquad l_0 > 0. \tag{31c}$$

We arrive at the main adaptive control result.

Theorem 2.7. *Let* $(A, B, C, D) \in \mathscr{S}$ *and* $\varphi \in \mathscr{N}$ *with Lipschitz constant* $\lambda > 0$. *Let* $r \in \mathbb{R}$ *be such that* $\varphi_r := [\mathbf{G}(1)]^{-1} r \in \operatorname{clos}(\operatorname{im} \varphi)$.

For all $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, *the solution* $n \mapsto (x_n, u_n, l_n)$ *of the initial-value problem* (31) *is such that assertions* (a)–(e) *of Theorem* 2.6 *hold. Moreover, if* $\varphi_r \in \operatorname{im} \varphi$ *is not a critical value of* φ, *then the non-increasing gain sequence* $(k_n) = (l_n^{-1})$ *converges to a positive value.*

Proof. Since (l_n) is monotone increasing, either $l_n \to \infty$ as $n \to \infty$ (Case (i)), or $l_n \to l^* \in (0, \infty)$ as $n \to \infty$ (Case (ii)). We consider these two cases separately.

Case (i). In this case, $k_n \downarrow 0$ as $n \to \infty$ and the hypotheses of Lemma 2.4 are satisfied. Therefore, (x_n) and $(\varphi(u_n))$ converge. In particular, the sequences (x_n) and $(\varphi(u_n))$ are bounded and so there exists a constant $\alpha > 0$ such that

$$k_n = \frac{1}{l_n} \geq \frac{1}{l_0 + \alpha n}, \qquad \forall n \in \mathbb{N}. \tag{32}$$

Therefore, assertions (a)–(e) of Theorem 2.6 hold.

Case (ii). In this case, $l_n \to l^* \in (0, \infty)$ as $n \to \infty$. Using (31c), we may conclude that the sequence $(r - Cx_n - D\varphi(u_n))$ is summable. This, together with (31b), implies that (u_n) is convergent to a finite limit, and hence $(\varphi(u_n))$ converges to a finite limit. By Lemma 2.5 it follows that assertions (a)–(e) of Theorem 2.6 hold.

Finally, assume that $\varphi_r \in \operatorname{im}\varphi$ is not a critical value. Seeking a contradiction, suppose that the monotone sequence (l_n) is unbounded (equivalently, $k_n \downarrow 0$ as $n \to \infty$). Then the hypotheses of Lemma 2.4 are satisfied and so (32) holds. By Theorem 2.6, $\varphi(u_n)$ converges to φ_r and $y_n = Cx_n + D\varphi(u_n)$ converges to r as $n \to \infty$; moreover, the convergence is of order ρ^{K_n} for some $\rho > 1$, where $K_n = \sum_{j=0}^{n} k_j$. Thus there exists constant $L_1 > 0$ such that

$$|r - y_n| \leq L_1 \rho^{-K_n}, \qquad \forall n \in \mathbb{N}. \tag{33}$$

Choose $\beta \geq \alpha$ such that $(\ln \rho)/\beta < 1$. By (32)

$$k_n \geq \frac{1}{l_0 + \beta n}, \qquad \forall n \in \mathbb{N}.$$

Hence

$$K_n \geq \sum_{j=0}^{n-1} \frac{1}{l_0 + \beta j} \geq \sum_{j=0}^{n-1} \int_j^{j+1} \frac{1}{l_0 + \beta t}\, dt = \int_0^n \frac{1}{l_0 + \beta t}\, dt = \ln\left[\left(\frac{l_0 + \beta n}{l_0}\right)^{1/\beta}\right]. \tag{34}$$

By (33),

$$l_{n+1} \leq l_n + L_1 \rho^{-K_n}, \qquad \forall n \in \mathbb{N},$$

and so, using (34),

$$l_n \leq l_0 + L_1 \sum_{j=0}^{n-1} \exp\left(\ln\left[\left(\frac{l_0 + \beta j}{l_0}\right)^{-\gamma}\right]\right) \leq l_0 + L_2 \sum_{j=0}^{n-1} (l_0 + \beta j)^{-\gamma}, \qquad \forall n \in \mathbb{N},$$

where $\gamma := (\ln \rho)/\beta \in (0, 1)$ and $L_2 := L_1 l_0^{\gamma}$. Therefore,

$$\begin{aligned} l_n &\leq l_0 + L_2 \left[l_0^{-\gamma} + \sum_{j=1}^{n-1} \int_{j-1}^{j} (l_0 + \beta t)^{-\gamma}\, dt \right] \\ &= l_0 + L_2 \left[l_0^{-\gamma} + \int_0^{n-1} (l_0 + \beta t)^{-\gamma}\, dt \right], \qquad \forall n \in \mathbb{N}, \end{aligned}$$

and we see that, for some suitable constant $L_3 > 0$,

$$l_n \leq L_3 (l_0 + \beta(n-1))^{1-\gamma} \leq L_3 (l_0 + \beta n)^{1-\gamma}, \qquad \forall n \in \mathbb{N}.$$

From this we may conclude that

$$-K_n \le -L_3^{-1}\sum_{j=0}^{n}(l_0+\beta j)^{\gamma-1} \le -L_3^{-1}\sum_{j=0}^{n-1}(l_0+\beta j)^{\gamma-1}$$
$$\le -L_3^{-1}\sum_{j=0}^{n-1}\int_j^{j+1}(l_0+\beta t)^{\gamma-1}\,dt,$$

and thus

$$-K_n \le -L_3^{-1}\int_0^n (l_0+\beta t)^{\gamma-1}\,dt = -a(l_0+\beta n)^{\gamma}+b,$$

with $a := 1/(L_3\beta\gamma) > 0$ and $b := al_0^{\gamma} > 0$. Therefore,

$$\rho^{-K_n} \le \rho^b \exp(-a(\ln\rho)(l_0+\beta n)^{\gamma}) = \rho^b \exp(-\delta(l_0+\beta n)^{\gamma}), \tag{35}$$

where $\delta := a\ln\rho > 0$. Using a change of variables it is readily verified that the non-increasing positive-valued function f defined by

$$f(t) = \rho^b \exp(-\delta(l_0+\beta t)^{\gamma}), \qquad \forall t \ge 0,$$

is in $L^1(\mathbb{R}_+,\mathbb{R})$. By the integral test for the convergence of an infinite series, it follows that the sequence $(f(n))$ is summable. By (35) we may now conclude that the sequence (ρ^{-K_n}) is summable. Combining this with (33) shows that (l_n) is bounded, contradicting the supposition of unboundedness of (l_n). ■

3. Adaptive Sampled-Data Low-Gain Control of Continuous-Time Systems

The next objective is to invoke Theorem 2.7 in the derivation of results on adaptive sampled-data integral control of continuous-time infinite-dimensional systems with input nonlinearities.

The generic system to be controlled is depicted schematically in Fig. 3, wherein $\varphi \in \mathcal{N}$ and $\mathbf{G}_c$ is the transfer function of a single-input, single-output, continuous-time, regular linear system Σ with state space X (a Hilbert space) and with generating operators (A_c, B_c, C_c, D_c). This means in particular that A_c generates a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\ge 0}$, $C_c \in \mathscr{B}(X_1,\mathbb{R})$ is an admissible observation operator for $\mathbf{T}$ and $B_c \in \mathscr{B}(\mathbb{R}, X_{-1})$ is an admissible control operator for $\mathbf{T}$. Here X_1 denotes the space $\mathrm{dom}(A_c)$ (the domain of A_c) endowed with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(s_0 I - A_c)^{-1}x\|$, where s_0 is any fixed element in the resolvent set of A_c. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ denote the norms on X_1 and X_{-1}, respectively. Then $X_1 \hookrightarrow X \hookrightarrow X_{-1}$ and $\mathbf{T}$ restricts (respectively,

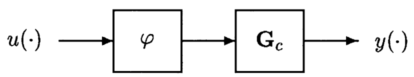

Fig. 3. The generic continuous-time system.

extends) to a strongly continuous semigroup on X_1 (respectively, X_{-1}). The exponential growth constant

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

is the same on all three spaces. The generator of $\mathbf{T}$ on X_{-1} is an extension of A_c to X (which is bounded as an operator from X to X_{-1}). We use the same symbol $\mathbf{T}$ (respectively, A_c) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A_c \in \mathscr{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A_c is X.

We regard a regular system Σ as synonomous with its generating operators and simply write $\Sigma = (A_c, B_c, C_c, D_c)$. The regular system is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable, that is, $\omega(\mathbf{T}) < 0$. The control operator B_c (respectively, observation operator C_c) is said to be *bounded* if $B_c \in \mathscr{B}(\mathbb{R}, X)$ (respectively, $C_c \in \mathscr{B}(X, \mathbb{R})$), otherwise, B_c (respectively, C_c) is said to be *unbounded*. In terms of the generating operators (A_c, B_c, C_c, D_c), the transfer function $\mathbf{G}_c(s)$ can be expressed as

$$\mathbf{G}_c(s) = C_{\mathrm{L}}(sI - A_c)^{-1} B_c + D_c,$$

where C_{L} denotes the so-called Lebesgue extension of C_c. The transfer function $\mathbf{G}_c(s)$ is bounded and holomorphic in any half-plane $\operatorname{Re} s > \alpha$ with $\alpha > \omega(\mathbf{T})$. Moreover,

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}_c(s) = D_c.$$

For any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, the state and output functions $x(\cdot)$ and $y(\cdot)$, respectively, satisfy the equations

$$\dot{x}(t) = A_c x(t) + B_c u(t), \qquad x(0) = x_0, \tag{36a}$$

$$y(t) = C_{\mathrm{L}} x(t) + D_c u(t) \tag{36b}$$

for almost all $t \geq 0$ (in particular $x(t) \in \operatorname{dom}(C_{\mathrm{L}})$ for almost all $t \geq 0$). The derivative on the left-hand side of (36a) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (36a) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, (36a) has unique strong solution (in the sense of Pazy [P, p. 109]) given by the variation of parameters formula

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B_c u(\tau)\, d\tau. \tag{37}$$

For more details on regular systems see [W1]–[W4]. For details on regular systems in the context of low-gain control the reader is referred to [LR], [LRT1] and [LT2].

For future reference we state the following lemma, the proof of which can be found in [LRT1].

Lemma 3.1. *Assume that* $\mathbf{T}$ *is exponentially stable and that* $B_c \in \mathscr{B}(\mathbb{R}, X_{-1})$ *is an admissible control operator for* $\mathbf{T}$. *If* $u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ *is such that* $\lim_{t\to\infty} u(t) = u_\infty$

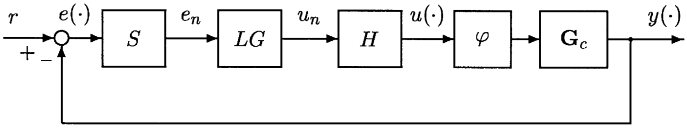

Fig. 4. Sampled-data low-gain control.

exists, then for all $x_0 \in X$ *the state* $x(\cdot)$ *given by* (37) *satisfies*

$$\lim_{t\to\infty} \|x(t) + A_c^{-1}B_c u_\infty\| = 0.$$

Let $\mathscr{R}_c$ denote the class of all continuous-time, single-input, single-output, exponentially stable, regular systems:

$$\mathscr{R}_c := \{\Sigma = (A_c, B_c, C_c, D_c) | \Sigma \text{ regular and exponentially stable}\}.$$

Let $\mathscr{S}_c \subset \mathscr{R}_c$ denote those systems of class $\mathscr{R}_c$ for which $\mathbf{G}_c(0)$ is positive:

$$\mathscr{S}_c := \{(A_c, B_c, C_c, D_c) \in \mathscr{R}_c | \mathbf{G}_c(0) = -C_{\mathrm{L}} A_c^{-1} B_c + D_c > 0\}.$$

For systems of class $\mathscr{S}_c$, our goal is to establish the efficacy of the control structure (a sampled-data analogue of (1)) depicted in Fig. 4, wherein S and H denote sampling and hold operations, respectively, and LG represents the (adaptive discrete-time) low-gain control law given by

$$u_{n+1} = u_n + l_n^{-1} e_n, \qquad u_0 \in \mathbb{R},$$
$$l_{n+1} = l_n + |e_n|, \qquad l_0 > 0.$$

With this in mind, we first construct a discretization of the continuous-time plant to which the results of the previous section may be applied. To this end, given a sequence $(u_n) \subset \mathbb{R}$, we define a continuous-time signal $u(\cdot)$ by the standard hold operation

$$u(t) = u_n, \qquad \text{for} \quad t \in [n\tau, (n+1)\tau), \quad n \in \mathbb{N}, \tag{38}$$

where $\tau > 0$ denotes the sampling period. If this signal is applied to the continuous-time system

$$\dot{x} = A_c x + B_c \varphi(u), \qquad x(0) = x_0 \in X, \tag{39a}$$
$$y = C_{\mathrm{L}} x + D_c \varphi(u), \tag{39b}$$

then, for all $n \in \mathbb{N}_0$, the state $x(n\tau + t)$ satisfies

$$x(n\tau + t) = \mathbf{T}_t x(n\tau) + (\mathbf{T}_t - I) A_c^{-1} B_c \varphi(u_n), \qquad \forall t \in [0, \tau). \tag{40}$$

Accordingly, we define $(x_n) \subset X$ by

$$x_n = x(n\tau), \qquad \forall n \in \mathbb{N}_0. \tag{41}$$

Clearly, $\mathbf{T}_\tau \in \mathscr{B}(X)$ and $(\mathbf{T}_\tau - I)A_c^{-1}B_c \in \mathscr{B}(\mathbb{R}, X)$ define appropriate state-space operators for the state evolution of the discretization. However, in general, regularity only guarantees that $y(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ so that, even with piecewise constant input functions, standard sampling of the output is not defined. Moreover, even if the output function is continuous (in which case standard sampling is defined), in general the resulting discrete-time system will not have a bounded observation operator. We therefore distinguish two cases: bounded and unbounded observation.

3.1. Bounded Observation

Assume that $C_c = C_{\mathrm{L}} \in \mathscr{B}(X, \mathbb{R})$. If $x_0 \in X$ and $u(\cdot)$ is given by (38), then the output $y(\cdot)$ given by (39b) is piecewise continuous, the discontinuities being at $n\tau$. It is clear that $y(\cdot)$ is right-continuous at $n\tau$ for all $n \in \mathbb{N}_0$. We define

$$y_n := y(n\tau), \qquad \forall n \in \mathbb{N}_0, \tag{42}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \mathbf{T}_\tau & (\mathbf{T}_\tau - I)A_c^{-1}B_c \\ C_c & D_c \end{pmatrix}. \tag{43}$$

The proof of the following proposition is an immediate consequence of Proposition 4.1 in [LT1].

Proposition 3.2. *Let* $(A_c, B_c, C_c, D_c) \in \mathscr{R}_c$ *with bounded observation operator* C_c. *Let* $\tau > 0$ *and* $(u_n) \subset \mathbb{R}$. *If* $u(\cdot)$ *given by* (38) *is applied to* (39), *then* (x_n) *and* (y_n) *given by* (41) *and* (42), *respectively, satisfy*

$$x_{n+1} = Ax_n + B\varphi(u_n), \tag{44a}$$

$$y_n = Cx_n + D\varphi(u_n), \tag{44b}$$

where (A, B, C, D) *is given by* (43). *Moreover,* A *is power-stable and, setting* $\mathbf{G}(\cdot) = C(\cdot I - A)^{-1}B + D$, *we have that*

$$\mathbf{G}(1) = C(I - A)^{-1}B + D = -C_cA_c^{-1}B_c + D_c = \mathbf{G}_c(0). \tag{45}$$

Consider the following adaptive sampled-data low-gain controller for (39)

$$u(t) = u_n, \qquad \text{for} \quad t \in [n\tau, (n+1)\tau), \quad n \in \mathbb{N}_0, \tag{46a}$$

$$y_n = y(n\tau), \qquad n \in \mathbb{N}_0, \tag{46b}$$

$$u_{n+1} = u_n + l_n^{-1}(r - y_n), \qquad u_0 \in \mathbb{R}, \tag{46c}$$

$$l_{n+1} = l_n + |r - y_n|, \qquad l_0 > 0. \tag{46d}$$

Theorem 3.3. *Let* $(A_c, B_c, C_c, D_c) \in \mathscr{S}_c$, $\varphi \in \mathscr{N}$ *with Lipschitz constant* $\lambda > 0$ *and* $\tau > 0$. *Assume that* C_c *is bounded. Let* $r \in \mathbb{R}$ *be such that* $\varphi_{cr} := [\mathbf{G}_c(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\varphi)$. *For all* $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, *the solution* $(x(\cdot), u(\cdot), (l_n))$ *of the closed-loop system given by* (39) *and* (46) *is such that the following assertions hold*:

(a) $\lim_{t\to\infty} \varphi(u(t)) = \varphi_{cr}$,
(b) $\lim_{t\to\infty} \|x(t) + A_c^{-1}B_c\varphi_{cr}\| = 0$,
(c) $\lim_{t\to\infty} y(t) = r$, *where* $y(\cdot)$ *is the output given by* (39b) *corresponding to the input function* $\varphi(u(\cdot))$ *with* $u(\cdot)$ *defined by* (46a),

(d) *if* $\varphi_{cr} \in \operatorname{im}\varphi$, *then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi_{cr})) = 0,$$

(e) *if* $\varphi_{cr} \in \operatorname{int}(\operatorname{im}\varphi)$, *then* $u(\cdot)$ *is bounded.*

Moreover, if $\varphi_{cr} \in \operatorname{im}\varphi$ *is not a critical value of* φ, *then the monotone gain sequence* $(k_n) = (l_n^{-1})$ *converges to a positive value.*

Remark. Statement (d) asserts that $u(t)$ converges as $t \to \infty$ if the set $\varphi^{-1}(\varphi_{cr})$ is a singleton, which, in turn, will be true if $\varphi_{cr} \in \operatorname{im}\varphi$ is not a critical value of φ.

Proof of Theorem 3.3. It follows from Proposition 3.2 that x_n (given by (41)), u_n and y_n satisfy (44) with (A, B, C, D) given by (43). By the same proposition, A is power-stable and $\mathbf{G}(1) = C(I - A)^{-1}B + D = \mathbf{G}_c(0) > 0$. Therefore, using Theorem 2.7, we see that

$$\lim_{n\to\infty} \varphi(u_n) = \varphi_r = [\mathbf{G}(1)]^{-1}r = [\mathbf{G}_c(0)]^{-1}r = \varphi_{cr},$$

and assertion (a) follows since the continuous-time signal $u(t)$ is given by (46a). Assertions (d) and (e) and the positivity of $\lim_{n\to\infty} k_n$ if $\varphi_{cr} \in \operatorname{im}\varphi$ is not a critical value of φ follow in the same way. Assertion (b) is a consequence of (a) and Lemma 3.1. Finally, assertion (c) follows from (a) and (b) and the boundedness of C_c. ■

3.2. Unbounded Observation

As mentioned earlier, in this case we cannot define a sampled output via (42). To overcome this difficulty we introduce a sampling operation which is a generalization of the simple averaging prototype

$$y_n = \frac{1}{\tau}\int_0^\tau y(n\tau + t)\, dt.$$

In the following, let $w \in L^2([0, \tau], \mathbb{R})$ be a function satisfying the conditions

$$\text{(a)} \quad \int_0^\tau w(t)\, d\tau = 1 \quad \text{and} \quad \text{(b)} \quad \int_0^\tau w(t)\mathbf{T}_t x\, dt \in X_1, \qquad \forall x \in X. \tag{47}$$

Whilst condition (47b) is difficult to check for general w, it is easy to show (using integration by parts) that (47b) holds if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = \tau$ such that $w|_{(t_{i-1}, t_i)} \in W^{1,1}((t_{i-1}, t_i), \mathbb{R})$ for $i = 1, 2, \ldots, m$.

We define a generalized sampling operation by

$$y_n = \int_0^\tau w(t) y(n\tau + t)\, dt, \qquad \forall n \in \mathbb{N}_0. \tag{48}$$

Introducing the linear operator

$$L : X \to X_1, \qquad x \mapsto \int_0^\tau w(t)\mathbf{T}_t x\, dt,$$

we define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \mathbf{T}_\tau & (\mathbf{T}_\tau - I)A_c^{-1}B_c \\ C_c L & C_c L A_c^{-1} B_c + \mathbf{G}_c(0) \end{pmatrix}. \tag{49}$$

By a routine application of the closed-graph theorem it follows that $L \in \mathscr{B}(X, X_1)$, and hence $C = C_c L \in \mathscr{B}(X, \mathbb{R})$.

Proposition 3.4. *Let* $(A_c, B_c, C_c, D_c) \in \mathscr{R}_c$. *Let* $\tau > 0$, *let* $w \in L^2([0,\tau], \mathbb{R})$ *be a function satisfying* (47) *and let* $(u_n) \subset \mathbb{R}$. *If* $u(\cdot)$ *given by* (38) *is applied to* (39), *then* (x_n) *and* (y_n) *given by* (41) *and* (48), *respectively, satisfy*

$$x_{n+1} = Ax_n + B\varphi(u_n), \tag{50a}$$

$$y_n = Cx_n + D\varphi(u_n), \tag{50b}$$

where (A, B, C, D) *is given by* (49). *Moreover,* A *is power-stable and setting* $\mathbf{G}(\cdot) = C(\cdot I - A)^{-1}B + D$ *we have that*

$$\mathbf{G}(1) = C(I - A)^{-1}B + D = -C_{\mathrm{L}}A_c^{-1}B_c + D_c = \mathbf{G}_c(0). \tag{51}$$

Proof. It is clear from (40) and (41) that (50a) holds. To derive (50b), we proceed as follows. For notational convenience, write $(v_n) = (\varphi(u_n))$. Let $n \in \mathbb{N}_0$. A straightforward computation shows that, for almost all $t \in [0, \tau)$,

$$\begin{aligned} y(n\tau + t) &= C_{\mathrm{L}}\mathbf{T}_t x_n + C_{\mathrm{L}} \int_0^t \mathbf{T}_s B_c v_n \, ds + D_c v_n \\ &= C_{\mathrm{L}}\mathbf{T}_t x_n + C_{\mathrm{L}}(\mathbf{T}_t - I)A_c^{-1}B_c v_n + D_c v_n. \end{aligned}$$

Therefore, using (47a) and (48), we obtain

$$y_n = \int_0^\tau w(t) C_{\mathrm{L}}\mathbf{T}_t x_n \, dt + \int_0^\tau w(t) C_{\mathrm{L}}\mathbf{T}_t A_c^{-1} B_c v_n \, dt + \mathbf{G}_c(0) v_n. \tag{52}$$

Choosing $(x_n^j)_{j\in\mathbb{N}} \subset X_1$ and $(b_n^j)_{j\in\mathbb{N}} \subset X$ such that

$$\lim_{j\to\infty} \|x_n - x_n^j\| = 0, \qquad \lim_{j\to\infty} \|B_c v_n - b_n^j\|_{-1} = 0, \tag{53}$$

we define $(y_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}$ by

$$y_n^j := \int_0^\tau w(t) C_{\mathrm{L}}\mathbf{T}_t x_n^j \, dt + \int_0^\tau w(t) C_{\mathrm{L}}\mathbf{T}_t A_c^{-1} b_n^j \, dt + \mathbf{G}_c(0) v_n, \qquad \forall j \in \mathbb{N}. \tag{54}$$

Since $x_n^j \in X_1$ and $b_n^j \in X$, this can be re-written as

$$y_n^j = C_c \int_0^\tau w(t)\mathbf{T}_t x_n^j \, dt + C_c \int_0^\tau w(t)\mathbf{T}_t A_c^{-1} b_n^j \, dt + \mathbf{G}_c(0) v_n, \qquad \forall j \in \mathbb{N},$$

and thus

$$y_n^j = Cx_n^j + CA_c^{-1} b_n^j + \mathbf{G}_c(0) v_n, \qquad \forall j \in \mathbb{N}. \tag{55}$$

Using (53) and the fact that C_c is an admissible output operator for the semigroup

$\mathbf{T}$ we have

$$\lim_{j\to\infty}\int_0^\tau |C_{\mathrm{L}}\mathbf{T}_t(x_n^j - x_n)|^2\,dt = 0, \qquad \lim_{j\to\infty}\int_0^\tau |C_{\mathrm{L}}\mathbf{T}_t(A_c^{-1}b_n^j - A_c^{-1}Bv_n)|^2\,dt = 0,$$

and, therefore, (52) and (54) yield

$$y_n = \lim_{j\to\infty} y_n^j.$$

Combining this with (55) and the boundedness of C, we obtain

$$y_n = Cx_n + CA_c^{-1}Bv_n + \mathbf{G}_c(0)v_n = Cx_n + Dv_n,$$

which is (50b). Moreover, since $\mathbf{T}$ is exponentially stable, it is clear that $A = \mathbf{T}_\tau$ is power-stable. Finally, to derive (51), note that

$$\begin{aligned}\mathbf{G}(1) &= C_c\int_0^\tau w(t)\mathbf{T}_t(I-\mathbf{T}_\tau)^{-1}(\mathbf{T}_\tau - I)A_c^{-1}B_c\,dt + CA_c^{-1}B_c + \mathbf{G}_c(0)\\ &= -CA_c^{-1}B_c + CA_c^{-1}B_c + \mathbf{G}_c(0)\\ &= \mathbf{G}_c(0).\end{aligned}$$

■

Consider the following adaptive sampled-data low-gain controller for (39):

$$u(t) = u_n, \qquad \text{for} \quad t \in [n\tau, (n+1)\tau), \quad n \in \mathbb{N}_0, \tag{56a}$$

$$y_n = \int_0^\tau w(t)y(n\tau + t)\,dt, \qquad n \in \mathbb{N}_0, \tag{56b}$$

$$u_{n+1} = u_n + l_n^{-1}(r - y_n), \qquad u_0 \in \mathbb{R}, \tag{56c}$$

$$l_{n+1} = l_n + |r - y_n|, \qquad l_0 > 0. \tag{56d}$$

We emphasize that (56) is a causal control strategy: y_n is available at a time $T \in (n\tau, (n+1)\tau]$ determined by the support of the function w.

Let $\mathscr{M}$ denote the space of finite signed Borel measures on $\mathbb{R}_+$ and let $\mathfrak{L}$ denote the Laplace transform. We now arrive at the main adaptive sampled-data control result in the unbounded observation case.

Theorem 3.5. *Let* $(A_c, B_c, C_c, D_c) \in \mathscr{S}_c$ *and* $\varphi \in \mathscr{N}$ *with Lipschitz constant* $\lambda > 0$. *Assume that* $\mathfrak{L}^{-1}(\mathbf{G}_c) \in \mathscr{M}$. *Let* $r \in \mathbb{R}$ *be such that* $\varphi_{cr} := [\mathbf{G}_c(0)]^{-1}r \in \operatorname{clos}(\operatorname{im}\varphi)$. *Let* $\tau > 0$ *and suppose that* $w \in L^2([0,\tau],\mathbb{R})$ *satisfies conditions* (47).

For all $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0,\infty)$, *the solution* $(x(\cdot), u(\cdot), (l_n))$ *of the closed-loop system given by* (39) *and* (56) *is such that the following assertions hold*:

(a) $\lim_{t\to\infty} \varphi(u(t)) = \varphi_{cr}$,
(b) $\lim_{t\to\infty} \|x(t) + A_c^{-1}B_c\varphi_{cr}\| = 0$,
(c) $\lim_{t\to\infty} |r - y(t) + C_{\mathrm{L}}\mathbf{T}_t x_0| = 0$, *where* $y(\cdot)$ *is the output given by* (39b) *corresponding to the input function* $\varphi(u(\cdot))$ *with* $u(\cdot)$ *defined by* (56a),
(d) *if* $\varphi_{cr} \in \operatorname{im}\varphi$, *then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi_{cr})) = 0,$$

(e) *if* $\varphi_{cr} \in \operatorname{int}(\operatorname{im}\varphi)$, *then* $u(\cdot)$ *is bounded.*

Moreover, if $\varphi_{cr} \in \operatorname{im}\varphi$ is not a critical value of φ, then the monotone gain sequence $(k_n) = (l_n^{-1})$ converges to a positive value.

Remarks. The assumption that $\mathfrak{L}^{-1}(\mathbf{G}_c) \in \mathscr{M}$ is not very restrictive and seems to be satisfied in all practical examples of exponentially stable systems. In particular, this assumption is satisfied if B_c or C_c is bounded. It follows from (c) that the error $e(t) = r - y(t)$ converges to 0 for all $x_0 \in \operatorname{dom}(A_c)$ since $C_{\mathrm{L}}\mathbf{T}_t x_0$ converges exponentially to 0 as $t \to \infty$ for all $x_0 \in \operatorname{dom}(A)$. If C_c is bounded, then this statement is true for all $x_0 \in X$. If C_c is unbounded and $x_0 \notin \operatorname{dom}(A_c)$, then $e(t)$ does not necessarily converge to 0 as $t \to \infty$. However, as the proof below shows, $e(t)$ is small for large t in the sense that $e(t) = e_1(t) + e_2(t)$, where the function e_1 is bounded with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2(\cdot)\exp(\alpha\cdot) \in L^2(\mathbb{R}_+, \mathbb{R})$ for some $\alpha > 0$.

Proof of Theorem 3.5. Using Proposition 3.4 instead of Proposition 3.2, all assertions, with the exception of (c), follow exactly as in the proof of Theorem 3.3 (which invokes Theorem 2.7). Due to the unboundeness of C_c we cannot use (b) in order to show that $y(t)$ converges to r as $t \to \infty$. However, we have

$$y(t) = C_{\mathrm{L}}\mathbf{T}_t x_0 + (\mathfrak{L}^{-1}(\mathbf{G}_c) \star \varphi(u))(t). \tag{57}$$

By assumption $\mathfrak{L}^{-1}(\mathbf{G}_c) \in \mathscr{M}$ and since $\lim_{t\to\infty} \varphi(u(t)) = \varphi_{cr}$ (by assertion (a)), it follows from Theorem 6.1, part (ii), p. 96 of [GLS] that

$$\lim_{t\to\infty} (\mathfrak{L}^{-1}(\mathbf{G}_c) \star \varphi(u))(t) = \mathbf{G}_c(0)\varphi_{cr} = r.$$

Combining this with (57) shows that assertion (c) holds. ∎

4. Example: Sampled-Data Control of a Diffusion Process with Output Delay

Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions) on the one-dimensional spatial domain $[0, 1]$, with scalar nonlinear pointwise control action applied at a point $x_b \in (0, 1)$, via a nonlinearity $\varphi \in \mathscr{N}$ with Lipschitz constant $\lambda > 0$:

$$z_t(t,x) = az_{xx}(t,x) + \delta(x - x_b)\varphi(u(t)), \qquad z(t,0) = 0 = z(t,1), \qquad \forall t > 0.$$

Furthermore, we assume a nonlinearity φ of saturation type, defined as follows:

$$u \mapsto \varphi(u) := \begin{cases} 1, & u \geq 1, \\ u, & u \in (0,1), \\ 0, & u \leq 0. \end{cases}$$

Cases of bounded and unbounded observation (with delay $h \geq 0$) will be treated separately. For simplicity, in each case we assume zero initial conditions:

$$z(t,x) = 0, \qquad \forall (t,x) \in [-h, 0] \times [0,1].$$

4.1. Bounded Observation

First, consider delayed scalar observation generated by a spatial averaging of the delayed state over an ε-neighbourhood of a point $x_c \in (0,1)$ with $x_c \geq x_b$:

$$y(t) = \frac{1}{2\varepsilon}\int_{x_c-\varepsilon}^{x_c+\varepsilon} z(t-h,x)\,dx.$$

With input $\varphi(u(\cdot))$ and output $y(\cdot)$, the process qualifies as a regular linear system with bounded observation and with the transfer function given by

$$\mathbf{G}_c(s) = \frac{e^{-sh}\sinh(x_b\sqrt{s/a})[\cosh((1-x_c+\varepsilon)\sqrt{s/a}) - \cosh((1-x_c-\varepsilon)\sqrt{s/a})]}{2\varepsilon s\sinh\sqrt{s/a}}.$$

Therefore, by Theorem 3.3, for each $(u_0, l_0) \in \mathbb{R} \times (0,\infty)$, the adaptive sampled-data control (with sampling at times $n\tau$, $\tau > 0$, and hold on intervals $[n\tau,(n+1)\tau)$) given by

$$\begin{aligned} u(t) &= u_n, \qquad t \in [n\tau,(n+1)\tau), \\ u_{n+1} &= u_n + k_n[r - y(n\tau)], \qquad k_n = 1/l_n, \\ l_{n+1} &= l_n + |r - y(n\tau)|, \end{aligned}$$

guarantees asymptotic tracking of every constant reference signal r satisfying

$$\frac{r}{\mathbf{G}_c(0)} = \frac{ar}{x_b(1-x_c)} \in \operatorname{clos}(\operatorname{im}\varphi).$$

For purposes of illustration, we adopt the following values:

$$a = 0.1, \quad x_b = \tfrac{1}{3}, \quad x_c = \tfrac{2}{3}, \quad h = 2, \quad \tau = 1, \quad \varepsilon = 0.01.$$

For reference signal $r = 0.75$, we have

$$\varphi_{cr} = \frac{r}{\mathbf{G}_c(0)} = \frac{0.75a}{x_b(1-x_c)} = 0.675 \in \operatorname{int}(\operatorname{im}\varphi) = (0,1).$$

In each of the following two cases,

(i) $l_0 = 2$,
(ii) $l_0 = 4$,

Figure 5 depicts the behaviour of the system under adaptive sampled-data control. This figure (and Fig. 6 below) was generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

4.2. Unbounded Observation

Next, we consider delayed pointwise scalar observation (output at point $x_c \in (0,1)$):

$$y(t) = z(t-h, x_c).$$

With input $\varphi(u(\cdot))$ and output $y(\cdot)$, the process now qualifies as a regular linear

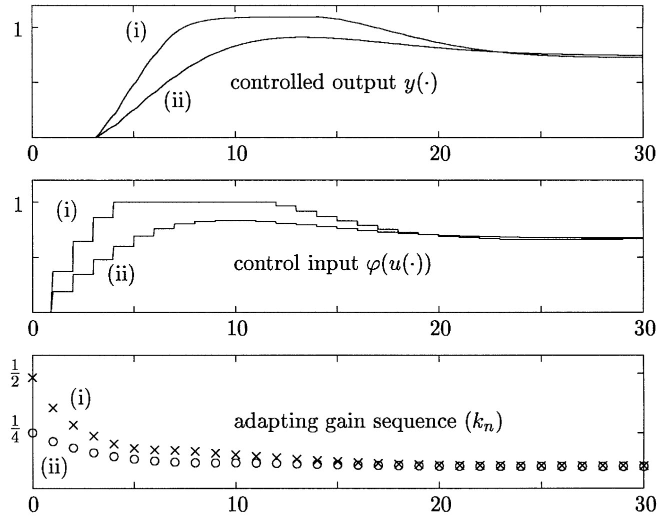

Fig. 5. Bounded observation: controlled output, control input and adapting gain.

system with unbounded observation and with the transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh}\sinh(x_b\sqrt{s/a})\sinh((1-x_c)\sqrt{s/a})}{a\sqrt{s/a}\sinh\sqrt{s/a}}.$$

As mentioned earlier, in this unbounded case we cannot define a sampled output via (42). Instead, we adopt the generalized sampling operation given by (48) with $w(\cdot) \equiv 1/\tau$:

$$y_n = \frac{1}{\tau}\int_0^\tau y(n\tau + t)\,dt.$$

Therefore, by Theorem 3.5, for each $(u_0, l_0) \in \mathbb{R} \times (0, \infty)$, the adaptive sampled-data control (with sampling at times $n\tau$, $\tau > 0$, and hold on intervals $[n\tau, (n+1)\tau))$ given by

$$\begin{aligned} u(t) &= u_n, \qquad t \in [n\tau, (n+1)\tau), \\ y_n &= \frac{1}{\tau}\int_0^\tau y(n\tau + t)\,dt, \\ u_{n+1} &= u_n + k_n[r - y_n], \qquad k_n = 1/l_n, \\ l_{n+1} &= l_n + |r - y_n|, \end{aligned}$$

guarantees asymptotic tracking of every constant reference signal r satisfying

$$\frac{r}{\mathbf{G}_c(0)} = \frac{ar}{x_b(1-x_c)} \in \operatorname{clos}(\operatorname{im}\varphi).$$

Again, for purposes of illustration, we adopt the following values:

$$a = 0.1, \quad x_b = \tfrac{1}{3}, \quad x_c = \tfrac{2}{3}, \quad h = 2, \quad \tau = 1.$$

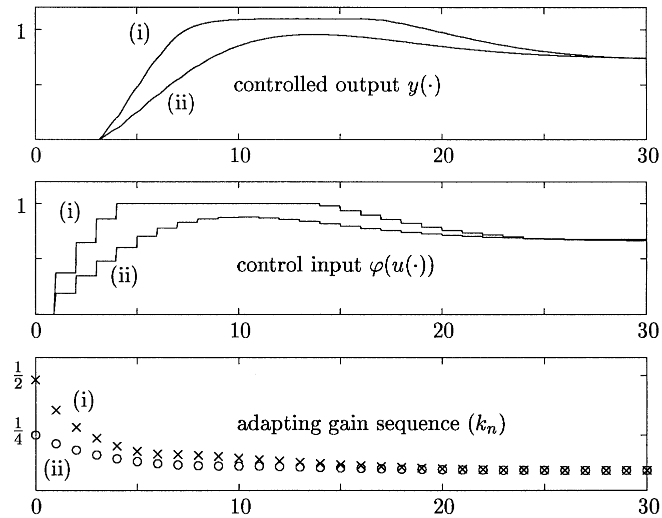

Fig. 6. Unbounded observation: controlled output, control input and adapting gain.

For reference signal $r = 0.75$, Fig. 6 depicts the behaviour of the system under adaptive sampled-data control in each of the two cases: (i) $l_0 = 2$, and (ii) $l_0 = 4$. Observe that, whilst the quantitative features differ, the qualitative features of the behaviour are akin to those of Fig. 5.

References

[C1] F. H. Clarke. *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[C2] P. A. Cook. Controllers with universal tracking properties, *Proc. Internat. IMA Conf. on Control: Modelling, Computation, Information*, Manchester, 1992.

[CSW] G. W. M. Coppus, S. L. Sha and R. K. Wood. Robust multivariable control of a binary distillation column, *Proc. IEE-D*, **130** (1983), 201–208.

[D] E. J. Davison. Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem, *IEEE Trans. Automat. Control*, **21** (1976), 35–47

[GLS] G. Gripenberg, S.-O. Londen and O. Staffans. *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

[LR] H. Logemann and E. P. Ryan. Time-varying and adaptive integral control of infinite-dimensional regular linear systems with input nonlinearities, *SIAM J. Control Optim.*, **38** (2000), 1120–1144.

[LRT1] H. Logemann, E. P. Ryan and S. Townley. Integral control of infinite-dimensional linear systems subject to input saturation, *SIAM J. Control Optim.*, **36** (1998), 1940–1961.

[LRT2] H. Logemann, E. P. Ryan and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain, *IEEE Trans. Automat. Control*, **44** (1999), 1315–1319.

[LT1] H. Logemann and S. Townley. Discrete-time low-gain control of uncertain infinite-dimensional systems, *IEEE Trans. Automat. Control*, **42** (1997), 22–37.

[LT2] H. Logemann and S. Townley. Low-gain control of uncertain regular linear systems, *SIAM J. Control Optim.*, **35** (1997), 78–116.

[LT3] H. Logemann and S. Townley. Adaptive integral control of time-delay systems, *Proc. IEE-D*, **144** (1997), 531–536.

[L1] J. Lunze. Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln*, **30** (1987), 2–6.

[L2] J. Lunze. *Robust Multivariable Feedback Control*, Prentice-Hall, London, 1988.

[MD1] D. E. Miller and E. J. Davison. The self-tuning robust servomechanism problem, *IEEE Trans. Automat. Control*, **34** (1989), 511–523.

[MD2] D. E. Miller and E. J. Davison. An adaptive tracking problem with a control input constraint, *Automatica*, **29** (1993), 877–887.

[M] M. Morari. Robust stability of systems with integral control, *IEEE Trans. Automat. Control*, **30** (1985), 574–577.

[P] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[W1] G. Weiss. Admissibility of unbounded control operators, *SIAM J. Control Optim.*, **27** (1989), 527–545.

[W2] G. Weiss. Admissible observation operators for linear semigroups, *Israel J. Math.*, **65** (1989), 17–43.

[W3] G. Weiss. The representation of regular linear systems on Hilbert spaces, pp. 401–416 in *Distributed Parameter System*, edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Basel, 1989.

[W4] G. Weiss. Transfer functions of regular linear systems, part I: characterization of regularity, *Trans. Amer. Math. Soc.*, **342** (1994), 827–854.